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- Etc.
  - Functional Programming & Type Theory prerequisite knowledge
  - You don't have to understand every detail: some of these lectures could fill a semester's worth of content.
  - Let us know if you're having any issues. This course should *not* be a source of stress
  - The more feedback you give us (and the more questions you ask), the better!

Intro to Type Theory and Lambda Calculus

Hype for Types

Jacob Neumann

14 January 2020

Table of Contents

Section 1

Functional Programming

Basic Idea of Functional Programming

- **Programming**: Feeding a computer strings of symbols to tell it to do stuff

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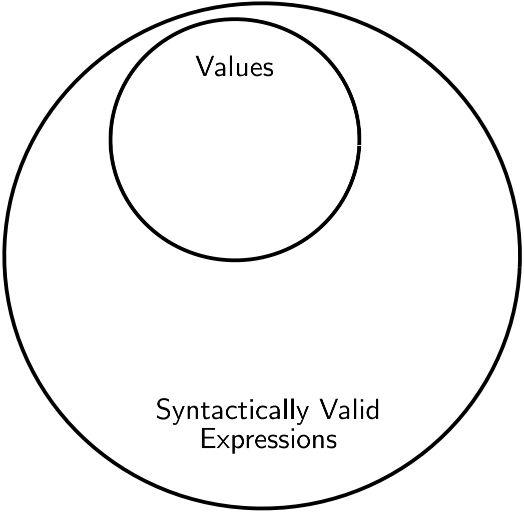
Basic Idea of Functional Programming



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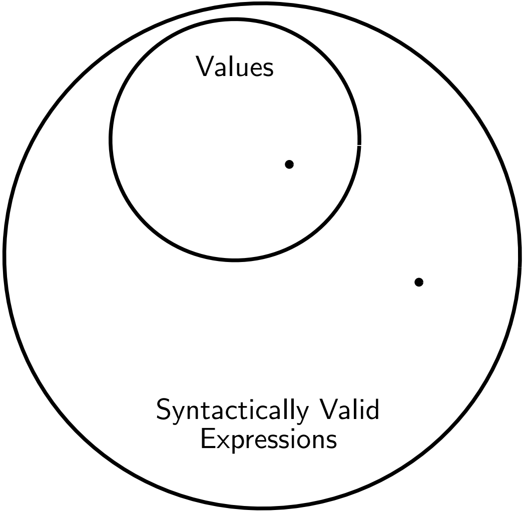

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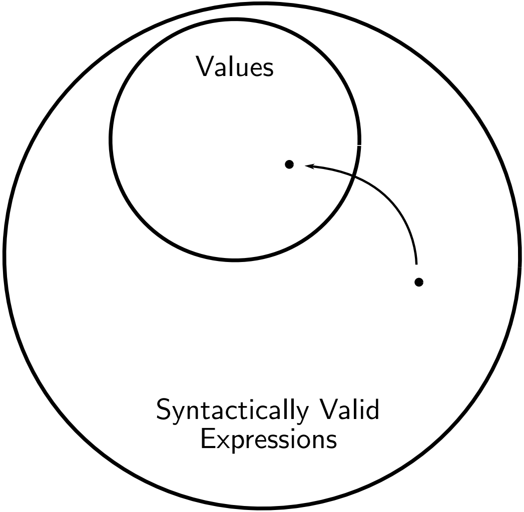

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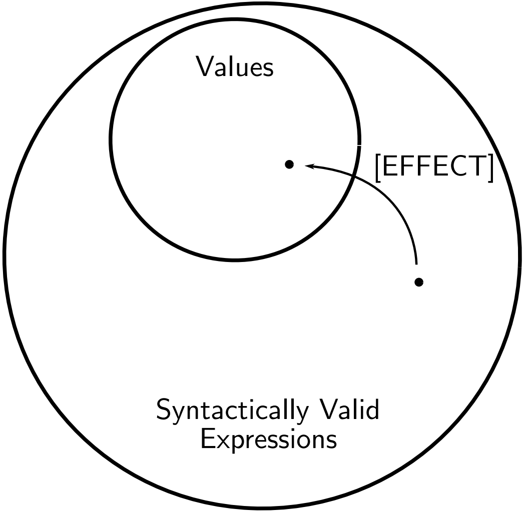

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- $0 \implies 0$

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- Bad effects: Injection attacks because of lack of input sanitization

Section 2

Ensuring Correctness

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- Disadvantages: Requires lots of effort and original thought, requires everyone be fluent in (nontrivial) mathematics, most code is too complex to be proven correct

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"If it compiles, it works"

The Central Dogma of Hype for Types

Push it to compile time!

Section 3

Type Theory

```
val x : int = 2
val y : int = 3
val b : bool = (x = y)
val s : string = if b
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RULE

if n is an integer, then
the expression n
is of type int

```
val x : int = 2
val y : int = 3
val b : bool =((x:int)= (y:int))
val s : string = if b
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RULE

if e1:t and e2:t, the expression e1=e2 has type bool

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val x : int = 2
val y : int = 3
val b : bool =((x:int)= (y:int))
val s : string = if (b:bool)
                 then ("good":string)
                 else ("bad":string)
```

RULE

if e:bool and e1:t and e2:t, then the expression if e then e1 else e2 has type t

Type Systems Have Two Faces

So, we specify a type system for our functional language by giving rules to determine the type of each expression.

Type Systems Have Two Faces

So, we specify a type system for our functional language by giving rules to determine the type of each expression.

This means there's two "sides" to the type system:

- The "practical" side: how the types guarantee features about how the code will evaluate (and how to design compilers that perform this typechecking)
- The "theoretical" side: the logical properties of the type system itself, and what the rules tell us about the relationships between types and expressions

The Theory Side

As an example of the theory side, let's look at the simplest typed functional programming language, the *simply-typed lambda calculus*.

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- Some basic types and expressions of those types
- A unit type
- Product types
- Function types

Section 4

The Simply-Typed Lambda Calculus

A Simple Type System

The simply-typed lambda calculus – true to its name – has a very simple type system. We define it inductively, with two base cases and two binary inductive type constructors.

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- There is a special type called **unit**
- If σ and τ are types, $\sigma * \tau$ is a type
- If σ and τ are types, $\sigma \to \tau$ is a type

Grammars

To state this concisely, we can give it as a *grammar*:

$$
\begin{aligned}
\sigma, \tau ::= \;& A && \text{(basic types)} \\
\mid\;& \textbf{unit} && \text{(unit)} \\
\mid\;& \sigma * \tau && \text{(product types)} \\
\mid\;& \sigma \to \tau && \text{(arrow types)}
\end{aligned}
$$

Contexts

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val x : int = 2
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As written, the variable `y` here gets bound to `4:int`.But if we replaced the first line with

```
val x : real = 3.0
```

then `y` would get bound to `6.0:real`. This is what we mean when we say that the type and value of an expression depend on the context.

Judgements and Contexts

For some syntactically-valid expression $\mathtt{x}$ and some type τ, we call this string of symbols

$$\mathtt{x} : \tau$$

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a "**typing judgement**". We read that as "$\mathtt{x}$ is of type τ".
A **context** Γ is just a finite list of typing judgements: the variables we've bound so far.

$$\Gamma = x_1 : \tau_1, \ldots, x_n : \tau_n$$

Terms-in-context

With this, we can specify how the terms of the lambda calculus are formed. Consider the following example:

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We say that $\mathtt{(x,x)}$ is a term of type $\tau * \tau$, in context $\mathtt{x} : \tau$. The lambda calculus is specified using these *terms-in-context*.

Example: Pairing

The following is a term-formation rule of the lambda calculus:

- If Γ is some context such that $\mathtt{x} : \sigma$ in context Γ and $\mathtt{y} : \tau$ in context Γ, then $\mathtt{(x,y)} : \sigma * \tau$ in context Γ.

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We indicate this more compactly using an **inference rule**.

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This gives an indication of how we'd recursively implement a lambda calculus typechecker: in order to verify that the expression $(\mathtt{x},\mathtt{y})$ is indeed of type $\sigma * \tau$, we just need to check that $\mathtt{x}$ is of type σ and $\mathtt{y}$ is of type τ.

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$$\frac{\Gamma \vdash p : \sigma * \tau}{\Gamma \vdash \mathsf{fst}(p) : \sigma} \qquad \frac{\Gamma \vdash p : \sigma * \tau}{\Gamma \vdash \mathsf{snd}(p) : \tau}$$

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$$\frac{\Gamma, x : \sigma \vdash e(x) : \tau}{\Gamma \vdash (\lambda x.e(x)) : \sigma \to \tau}$$

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$$\frac{\Gamma \vdash f : \sigma \to \tau \qquad \Gamma \vdash t : \sigma}{\Gamma \vdash (ft) : \tau}$$

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- What if we had a way of proving (in a way that could be verified by the typechecker) that our code must meet a certain spec? (Interactive Theorem Proving)

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So stay tuned!

Thank you!