$$
\begin{tikzcd}
 & (M \times M) \times M \arrow[rr, dash] \arrow[ddl, dash] & & M \times (M \times M) \arrow[ddr, dash] & \\
 & 1 \times M \arrow[r, dash] \arrow[ddr, dash] & M \times M \arrow[r, dash] \arrow[dd, dash] & M \times 1 \arrow[ddl, dash] & \\
M \times M \arrow[drr, dash] & & & & M \times M \arrow[dll, dash] \\
 & & M & &
\end{tikzcd}
$$

Monoid and Group Objects

A Taste of Algebraic Theories in Categorical Logic

Jacob Neumann

80-413/713 - November 2020

Overview

- Study classical algebraic structures with the tools of category theory
- Study categorical structures in an algebraic framework

Some sources

- Chaps. 4 and 10 of textbook (might cover Chap. 10 in lecture)
- Categorical Logic notes by Awodey & Bauer
- Various nLab pages (group object, monoid in a monoidal category, Eckmann-Hilton argument, endofunctor, Lawvere Theory, variety of algebras,[1] internalization, etc.)

[1]Not "algebraic varieties" – that's something different.

0 Monoids and Categories

Monoids so far...

(Classical definition) A **monoid** consists of a set M, a binary operation $\cdot : M \times M \to M$, and an element $e \in M$ such that

- For all $x, y, z \in M$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- For all $x \in M$, $e \cdot x = x = x \cdot e$

Monoids so far. . .

- We have seen the **category of monoids**, Mon, whose objects are monoids and whose morphisms are monoid homomorphisms
- We also saw that we could view **monoids as single-object categories**, where the morphisms corresponded to the elements of the monoid, and composition the binary operation
- And now. . . **monoids in a category**

Internalization

A monoid consists of a set M, a function $\cdot : M \times M \to M$, and an element $e \in M$ such that...

A monoid consists of an **object** M, a **morphism** $\cdot : M \times M \to M$, and a **morphism** $e : 1 \to M$ such that...

Monoid Object

Given a category $\mathbb{C}$ with all finite products, a **monoid in $\mathbb{C}$** consists of

- an object M of $\mathbb{C}$
- a morphism $\mu : M \times M \to M$
- a morphism $e : 1 \to M$

$$\begin{tikzcd}
M \times M \arrow[dr, "\mu"'] & 1 \arrow[d, "e"] \\
& M
\end{tikzcd}$$

such that...

- μ is associative
- e is a unit for μ

Monoid Object

$$\begin{tikzcd}[column sep=small]
& (M \times M) \times M \arrow[rr, leftrightarrow] \arrow[dl, "\mu \times 1_M"'] & & M \times (M \times M) \arrow[dr, "1_M \times \mu"] & \\
M \times M \arrow[rr, "\mu"'] & & M & & M \times M \arrow[ll, "\mu"]
\end{tikzcd}$$

$$\begin{tikzcd}
1 \times M \arrow[r, "e \times 1_M"] \arrow[dr] & M \times M \arrow[d, "\mu"] & M \times 1 \arrow[l, "1_M \times e"'] \arrow[dl] \\
& M &
\end{tikzcd}$$

Basic Examples

- Classical monoids: monoids in Set
- Ordered monoids: monoids in Pos

$$\begin{tikzcd}
(\mathbb{R} \times \mathbb{R}, \leq) \arrow[dr, "+"'] & 1 \arrow[d, "0"] \\
& (\mathbb{R}, \leq)
\end{tikzcd}$$

$$(a, b) \leq (c, d) \quad \overset{\text{def}}{\iff} \quad a \leq c \text{ and } b \leq d \quad \implies \quad a + b \leq c + d$$

More Examples

- If a poset $(P, \leq)$ has finite meets and a maximal element, then this is a monoid in the category of posets:
  - $\wedge : P \times P \to P$ such that $(p, q) \leq (r, s) \implies p \wedge q \leq r \wedge s$
  - $\top : 1 \to P$
  - $p \wedge (q \wedge r) = (p \wedge q) \wedge r$
  - $p \wedge \top = p = \top \wedge p$
- A monoid in Group, the category of groups and group homomorphisms, is a ring.
- Comonoids in $\mathbb{C}$ are monoids in $\mathbb{C}^{\text{op}}$:
  - $\mu : M \to M \times M$
  - $e : M \to 1$

Monoids in Presheaves

Recall that for any $\mathbb{C}$, the category $\hat{\mathbb{C}} = \text{Hom}(\mathbb{C}^{\text{op}}, \text{Set})$ of presheaves on $\mathbb{C}$ is cartesian:

- Terminal object of $\hat{\mathbb{C}}$ is the constant functor $\Delta 1$ (which sends each object to $1 = \{\star\}$ and each morphism to 1_1)
- For $F, G : \mathbb{C}^{\text{op}} \to \text{Set}$, define $F \times G$ to be the presheaf sending x to $F(x) \times G(x)$ and $u : x \to y$ to $(Fu \times Gu) : (Fy \times Gy) \to (Fx \times Gx)$

$$\begin{tikzcd} F \times F \arrow[dr, "\mu"'] & \Delta 1 \arrow[d, "e"] \\ & F \end{tikzcd}$$

Monoids in Presheaves

$$\begin{tikzcd} F \times F \arrow[dr, "\mu"'] & \Delta 1 \arrow[d, "e"] \\ & F \end{tikzcd}$$

- For each object x of $\mathbb{C}$,

$$\mu_x : (Fx) \times (Fx) \to Fx$$
$$e_x : 1 \to Fx$$

- Naturality: for all $u : x \to y$ in $\mathbb{C}$

$$\mu_x \circ (F(u) \times F(u)) = F(u) \circ \mu_y \qquad e_x = F(u) \circ e_y$$

Some Examples

- A monoid M in $\mathsf{Set}^{[1]^{\mathsf{op}}}$ is a monoid homomorphism $M(u)$ between the monoids $(M1, \mu_1, e_1)$ and $(M0, \mu_0, e_0)$.
- A monoid M in $\mathsf{Set}^{I^{\mathsf{op}}}$ for any set I viewed as a discrete category is just an I-indexed family of monoids $\{(M(i), \mu_i, e_i)\}_{i \in I}$
- A monoid M in $\mathsf{Set}^{(\mathbb{N}, \leq)} = \mathsf{Set}^{(\mathbb{N}, \geq)^{\mathsf{op}}}$ is a sequence $\{(Mi, \mu_i, e_i)\}_{i \in \mathbb{N}}$ of monoids with monoid homomorphisms $f_i : Mi \to M(i+1)$

1 Groups and Categories

Groups (classical definition)

A **group** consists of a set G, a binary operation $\cdot : G \times G \to G$, an element $e \in G$, and an operation $(-)^{-1} : G \to G$ such that

- For all $x, y, z \in G$, $(x \cdot y) \cdot z = x \cdot (y \cdot z)$
- For all $x \in G$, $e \cdot x = x = x \cdot e$
- For all $x \in G$, $x \cdot x^{-1} = e = x^{-1} \cdot x$

Equivalently: A group is a monoid where every element has an inverse

Category of Groups

A **group homomorphism** from G to H is a function $f : G \to H$ such that

$$f(x \cdot_G y) = f(x) \cdot_H f(y) \qquad \text{for all } x, y \in G$$

Exercise The category, Group, of groups and group homomorphisms is a full subcategory of Mon.

Groups as Categories

We also had a way of viewing groups as singleton category. Equivalently:

- A group is a monoid (viewed as a singleton category) where all morphisms are isomorphisms
- A group is a groupoid with one object

Group Object

Given a category $\mathbb{C}$ with all finite products, a **group in $\mathbb{C}$** consists of

- an object G of $\mathbb{C}$
- a morphism $\mu : G \times G \to G$
- a morphism $e : 1 \to G$
- a morphism $i : G \to G$

$$\begin{tikzcd}
G \times G \arrow[dr, "\mu"'] & 1 \arrow[d, "e"] & G \arrow[dl, "i"] \\
& G &
\end{tikzcd}$$

Group in a category

$$
\begin{tikzcd}
 & (G \times G) \times G \arrow[rr, leftrightarrow] \arrow[dl, "\mu \times 1_G"'] & & G \times (G \times G) \arrow[dr, "1_G \times \mu"] & \\
G \times G \arrow[drr, "\mu"'] & & & & G \times G \arrow[dll, "\mu"] \\
 & & G & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
1 \times G \arrow[r, "e \times 1_G"] \arrow[dr] & G \times G \arrow[d, "\mu"] & G \times 1 \arrow[l, "1_G \times e"'] \arrow[dl] \\
 & G &
\end{tikzcd}
$$

Group in a category

$$\begin{tikzcd}
G \times G \arrow[d, "1_G \times i"'] & G \arrow[l, "\langle 1_G, 1_G \rangle"'] \arrow[r, "\langle 1_G, 1_G \rangle"] \arrow[d, "e \circ !_G"] & G \times G \arrow[d, "i \times 1_G"] \\
G \times G \arrow[r, "\mu"'] & G & G \times G \arrow[l, "\mu"]
\end{tikzcd}$$

Basic Examples

- Groups in Set: classical groups
- Groups in Top: topological groups

$$\mu : X \times X \to X$$

continuous with respect to the topology on X and the product topology on $X \times X$, and

$$(-)^{-1} : X \to X$$

continuous.

Functor Group

If $\mathbb{C}, \mathbb{D}$ are categories and $\mathbb{D}$ has all finite products, then the functor category $\mathbb{D}^{\mathbb{C}}$ has finite products:

- $(F \times G)(x) = Fx \times Gx$
- $(F \times G)(u : x \to y) = (Fu \times Gu) : (Fx \times Gx) \to (Fy \times Gy)$

A group in $\mathbb{D}^{\mathbb{C}}$ consists of

- A functor F
  - For each $x \in \mathbb{C}$, a set Fx
  - For each $u : x \to y$ in $\mathbb{C}$, a function $Fu : Fx \to Fy$
- For each $x \in \mathbb{C}$, a function $\mu_x : Fx \times Fx \to Fx$
- For each $x \in \mathbb{C}$, an element $e_x \in Fx$
- For each $x \in \mathbb{C}$, an operation $i_x : Fx \to Fx$

Morphisms on Two Levels

- If (F, μ, e, i) is a group in $\mathbb{D}^{\mathbb{C}}$ and $u : x \to y$ is a morphism in $\mathbb{C}$, then $F(u) : F(x) \to F(y)$ is a group homomorphism between the groups (Fx, μ_x, e_x, i_x) and (Fy, μ_y, e_y, i_y):

$$\begin{tikzcd} Fx \times Fx \arrow[r, "\mu_x"] \arrow[d, "Fu \times Fu"'] & Fx \arrow[d, "F(u)"] \\ Fy \times Fy \arrow[r, "\mu_y"'] & Fy \end{tikzcd}$$

(also $e_y = F(u) \circ e_x$ and $i_y \circ F(u) = F(u) \circ i_x$)

Morphisms on Two Levels

- If (F, μ, e, i) and (F', μ', e', i') are groups in $\mathbb{D}^{\mathbb{C}}$ and $\theta : F \to F'$ a natural transformation such that

$$\begin{aligned} \mu' \circ (\theta \times \theta) &= \theta \circ \mu \\ e' &= \theta \circ e \\ i' \circ \theta &= \theta \circ i \end{aligned}$$

  Then each component $\theta_x : F(x) \to F'(x)$ is a group homomorphism.

2 Some Theory

Category of groups in a category

Let $\mathbb{C}$ be any cartesian category. Define $\mathsf{Group}(\mathbb{C})$ to be the category

- Whose objects are groups (G, μ, e, i) in $\mathbb{C}$
- Whose morphisms $f : (G, \mu, e, i) \to (G', \mu', e', i')$ are morphisms $f : G \to G'$ in $\mathbb{C}$ which commute with the group structure:

$$\begin{array}{ccc} G \times G & \xrightarrow{\mu} & G \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ G' \times G' & \xrightarrow[\mu']{} & G' \end{array} \qquad \begin{array}{ccc} 1 & \xrightarrow{e} & G \\ {\scriptstyle 1_1}\downarrow & & \downarrow{\scriptstyle f} \\ 1 & \xrightarrow[e']{} & G' \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{i} & G \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f} \\ G' & \xrightarrow[i']{} & G' \end{array}$$

$\mathsf{Group} = \mathsf{Group}(\mathsf{Set})$

The evaluation functor

$\mathsf{Group}\left(\mathsf{Set}^{\mathbb{C}}\right) = \,?$

A minute ago, we saw that for any $(F, \mu, e, i) \in \mathsf{Group}\left(\mathbb{D}^{\mathbb{C}}\right)$ and any $x \in \mathbb{C}$, the object $Fx \in \mathbb{D}$ had a group structure on it given by μ_x, e_x and i_x.

$$\begin{aligned}
\mathsf{ev} : \mathsf{Group}(\mathsf{Set}^{\mathbb{C}}) &\to \mathsf{Group}^{\mathbb{C}} \\
\mathsf{ev}(F) : \mathbb{C} &\to \mathsf{Group} \\
&: x \mapsto \mathsf{ev}(F, x) = (Fx, \mu_x, e_x, i_x) \\
&: (u : x \to y) \mapsto Fu \in \mathsf{Hom}_{\mathsf{Group}}(Fx, Fy) \\
\mathsf{ev}(\theta : F \to F') &: \mathsf{ev}(F) \to \mathsf{ev}(F') \\
\mathsf{ev}(\theta)_x &= \theta_x \in \mathsf{Hom}_{\mathsf{Group}}(Fx, F'x)
\end{aligned}$$

When life gives you lemmas. . .

Lemma ev is faithful

Lemma ev is full

Lemma ev is a bijection on objects

$$\mathrm{Group}\left(\mathrm{Set}^{\mathbb{C}}\right) \cong \mathrm{Group}^{\mathbb{C}}$$
$$\mathrm{Group}\left(\hat{\mathbb{C}}\right) \cong \mathrm{Group}^{\mathbb{C}^{\mathrm{op}}}$$
$$\mathrm{Mon}\left(\mathrm{Set}^{\mathbb{C}}\right) \cong \mathrm{Mon}^{\mathbb{C}}$$

"A group in presheaves is a presheaf of groups"

Monoids in Mon? Groups in Group?

What about Mon(Mon) or Group(Group)?

$$\begin{tikzcd}
M \times M \arrow[drr, "(-\cdot-)"'] & 1 \arrow[dr, "e_\cdot"] & & 1 \arrow[dl, "e_\star"'] & M \times M \arrow[dll, "(-\star-)"] \\
& & (M, \cdot, e_\cdot) & &
\end{tikzcd}$$

$$(w \cdot x) \star (y \cdot z) = (w \star y) \cdot (x \star z)$$

The Eckmann-Hilton Argument

Thm If $((M, \cdot, e_\cdot), \star, e_\star)$ is a monoid in Mon, then $e_\cdot = e_\star$, $\cdot$ and $\star$ are equal, and $(M, \cdot, e_\cdot)$ is commutative:

$$x \cdot y = y \cdot x \qquad \text{for all } x, y \in M.$$

Proof

- $e_\star : \{*\} \to (M, \cdot, e_\cdot)$ must be a monoid homomorphism, so $e_\star = e_\cdot$.

The Eckmann-Hilton Argument

Proof (cont.)

- $e_\star : \{*\} \to (M, \cdot, e_\cdot)$ must be a monoid homomorphism, so $e_\star = e_\cdot$. Write $e = e_\cdot = e_\star$.
- Then, for any $x, y \in M$

$$\begin{aligned} x \cdot y &= (x \star e) \cdot (e \star y) && (e = e_\star) \\ &= (x \cdot e) \star (e \cdot y) && (\star \text{ is a homomorphism}) \\ &= x \star y \end{aligned}$$

- For any $x, y \in M$,

$$x \cdot y = (1 \cdot x) \cdot (y \cdot 1) = (1 \cdot y) \cdot (x \cdot 1) = y \cdot x$$

Result

Defn. A monoid (resp. group) X is said to be **commutative** (resp. **abelian**) if for all $x, y \in X$, $x \cdot y = y \cdot x$. The full subcategory of commutative monoids (resp. abelian groups) is denoted CMon (resp. Ab).

- $\mathsf{Mon}(\mathsf{Mon}) \cong \mathsf{CMon}$
- $\mathsf{Group}(\mathsf{Group}) \cong \mathsf{Ab}$
- $\mathsf{Mon}(\mathsf{CMon}) \cong \mathsf{CMon}$
- $\mathsf{Group}(\mathsf{Ab}) \cong \mathsf{Ab}$

3 Generalizations

If it looks like a functor and smells like a functor...

- Monoid objects and group objects are defined by a diagram
- Diagrams are functors

$$
\begin{tikzcd}
 & (M \times M) \times M \arrow[rr, leftrightarrow] \arrow[ddl] & & M \times (M \times M) \arrow[ddr] & \\
 & 1 \times M \arrow[r] \arrow[ddr] & M \times M \arrow[dd] & M \times 1 \arrow[l] \arrow[ddl] & \\
M \times M \arrow[drr] & & & & M \times M \arrow[dll] \\
 & & M & &
\end{tikzcd}
$$

Functorial Semantics

An **algebraic theory** $\mathbb{T}$ specifies some collection of data (sets, constants, functions, etc.) and some equations. For instance, the theory of abelian groups:

- Four pieces of data (G, μ, e, and i)
- Four equations (the associativity, identity, inverse, and commutative laws).

A **model** of $\mathbb{T}$ in a finite product category $\mathbb{C}$ consists of objects & morphisms of $\mathbb{C}$ interpreting $\mathbb{T}$, such that all of $\mathbb{T}$'s equations come out as true. For instance, a model of $\mathbb{T}_{\mathsf{Mon}}$ (the theory of monoids) in $\mathbb{C}$ is a monoid object in $\mathbb{C}$.

Functorial Semantics

For any algebraic theory $\mathbb{T}$, there is a finite product category $\mathcal{C}_{\mathbb{T}}$ called the **syntactic category of $\mathbb{T}$** such that

$$\mathsf{Mod}\,(\mathbb{T}, \mathbb{C}) \cong \mathsf{Hom}_{\mathsf{FP}}(\mathcal{C}_{\mathbb{T}}, \mathbb{C})$$

Thm The following are equivalent:

1. For every model M of $\mathbb{T}$ in $\mathbb{C}$, $M \models s = t$
2. $\mathbb{T} \vdash s = t$

To learn more about functorial semantics,
take categorical logic!

Monoidal Categories

Another possible generalization: we can define monoid objects in categories which do not have finite products.

A category $\mathbb{C}$ is called **monoidal** if it is equipped with a functor $\otimes : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $I \in \mathbb{C}$ which "behave like $\times$ and 1":

- $X \otimes (Y \otimes Z) \cong (X \otimes Y) \otimes Z$
- $X \otimes I \cong X$
- $\ldots$

Monoidal monoid

$$
\begin{tikzcd}
 & (M \otimes M) \otimes M \arrow[rr, leftrightarrow] \arrow[ddl, "\mu \otimes 1_M"'] & & M \otimes (M \otimes M) \arrow[ddr, "1_M \otimes \mu"] & \\
 & I \otimes M \arrow[r, "e \otimes 1_M"] \arrow[ddr] & M \otimes M \arrow[dd, "\mu"] & M \otimes I \arrow[l, "1_M \otimes e"'] \arrow[ddl] & \\
M \otimes M \arrow[drr, "\mu"'] & & & & M \otimes M \arrow[dll, "\mu"] \\
 & & M & &
\end{tikzcd}
$$

Example: the category of endofunctors

The category $\mathrm{End}(\mathbb{C}) = \mathbb{C}^{\mathbb{C}}$ does not – in general – have finite products.
But it is monoidal under composition, with unit $\mathrm{id}_{\mathbb{C}}$:

- $F \circ (G \circ H) = (F \circ G) \circ H$ for all $F, G, H : \mathbb{C} \to \mathbb{C}$
- $F \circ \mathrm{id}_{\mathbb{C}} = F$
- ...

Monoids in the category of endofunctors

Claim Every adjunction $\begin{tikzcd} \mathbb{C} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{D} \arrow[l, bend left, "U"] \end{tikzcd}$ gives rise to a **monad** (a monoid in $\mathrm{End}(\mathbb{C})$)

- The object is $UF \in \mathrm{End}(\mathbb{C})$
- For each $C \in \mathbb{C}$, we have the counit $\epsilon_{FC} : FUFC \to FC$ and thus

$$U(\epsilon_{F(-)}) : UF \circ UF \to UF$$

- The unit of the adjunction:

$$\eta : \mathrm{id}_{\mathbb{C}} \to UF$$

- Can check that the monoid laws are satisfied

Example of a monad: List

If F : Set $\to$ Mon is the free monoid functor, and U : Mon $\to$ Set is its right adjoint, the forgetful functor, we'll write List for the composition $U \circ F$.

$$\mathsf{List}(A) = \{[a_1, \ldots, a_n] \ : \ n \in \mathbb{N} \text{ and } a_1, \ldots, a_n \in A\}$$

- μ_A : $\mathsf{List}(\mathsf{List}(A)) \to \mathsf{List}(A)$ concatenates lists of lists:

$$\mu_{\mathbb{Z}}[[1,2,3],[\,],[4],[5,6]] = [1,2,3,4,5,6]$$

- η_A : $A \to \mathsf{List}(A)$ creates singleton lists

$$\eta_{\mathbb{Z}}(4) = [4]$$

Thank you!