Category Theory (80-413/713) F20 HW9, Exercise 5 Solution

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Problem:

> Let I be a small category, and $\Delta : \mathsf{Set} \to \mathsf{Set}^I$ be the constant diagram functor. Prove that Δ is fully faithful iff $\operatorname{colim}_i \Delta 1 = 1$.

Solution:

1

First, note the form of this problem: we have two statements – (a) Δ is fully faithful, and (b) $\operatorname{colim}_i \Delta 1 = 1$ – which we're proving equivalent. As we'll see going through the proof, (a) appears much stronger than (b), and the proof of (a) $\Longrightarrow$ (b) will be quick. Indeed, we'll see that (a) is a universal statement, and (b) just a single instance of (a). The difficulty of this proof is showing that (b) – which is a "local" property of just one particular set, 1 – is "universal" in the appropriate sense. Specifically, we'll have to show that $\operatorname{colim}_i \Delta 1 = 1$ implies $\epsilon_X : \operatorname{colim}_i \Delta X \cong X$ *for all* X.

2

Let's dispense with (a) $\Longrightarrow$ (b). Recall from lecture:

> **Fact 1** U is fully faithful iff ϵ is a natural iso (For any adjunction $F \dashv U$)

Therefore, since Δ is a fully faithful right adjoint, the counit $\epsilon_X : \operatorname{colim}_i \Delta X \to X$ is an isomorphism (that is, a bijection) for all sets X. So, in particular,

$$\epsilon_1 : \operatorname*{colim}_i \Delta 1 \xrightarrow{\sim} 1.$$

Conclude that $\operatorname{colim}_i \Delta 1$ is a singleton set, i.e. a terminal object of Set. As usual, we suppress the distinction between uniquely isomorphic objects, so

$$\operatorname*{colim}_i \Delta 1 = 1.$$

3

(b) $\Longrightarrow$ (a) is significantly more involved. Start with the naturality of ϵ: for all functions $f : A \to B$, the square

$$\begin{array}{ccc} \operatorname{colim}_i \Delta A & \xrightarrow{\epsilon_A} & A \\ {\scriptstyle \operatorname{colim}_i \Delta(f)} \downarrow & & \downarrow {\scriptstyle f} \\ \operatorname{colim}_i \Delta B & \xrightarrow[\epsilon_B]{} & B \end{array}$$

commutes. **Recall 1** that we define $\operatorname{colim}_i \Delta(f)$ by taking the maps

$$\Delta(A)(j) \xrightarrow{(\Delta f)_j} \Delta(B)(j) \xrightarrow{\mathsf{inc}_j^{\Delta B}} \operatorname{colim}_i \Delta B$$

for each object j of I, and then combining them together using the **"co-pairing"**

operation of colimits: 2

$$\left[\mathsf{inc}_j^{\Delta B} \circ (\Delta f)_j \;\middle|\; j \in I\right] : \operatorname*{colim}_i \Delta A \to \operatorname*{colim}_i \Delta B.$$

But $\Delta(A)(j)$ is just A by definition of Δ, and likewise for B. And the j-component of the natural transform $\Delta f : \Delta A \to \Delta B$ is just f, for each j. So we can state the naturality of ϵ as

Nat. ϵ $\quad f \circ \epsilon_A = \epsilon_B \circ [\mathsf{inc}_j^{\Delta B} \circ f]$ (For all $f : A \to B$)

4

Let's combine this with a **useful fact about coproducts in Set.** 3 Specifically this:

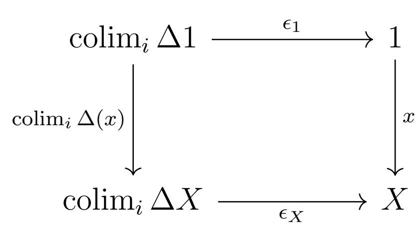
Fact 2 $\quad \coprod_{x \in X} 1 = X$ (For all sets X)

If you spell out the details of this fact, you'll see that it basically involves defining another constant functor $\Delta_X : \mathsf{Set} \to \mathsf{Set}^X$ where $\Delta_X(Y)(x) = Y$ for all $x \in X$ and all sets Y, and then proving that X satisfies the universal mapping property of the colimit over the diagram $\Delta_X(1)$. Note that for each $x \in X$ the inclusion map

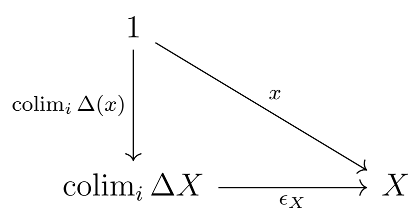
$$\mathsf{inc}_x^{\Delta_X(1)} : \Delta_X(1)(x) \to X$$

is just the **element** $x : 1 \to X$. 4

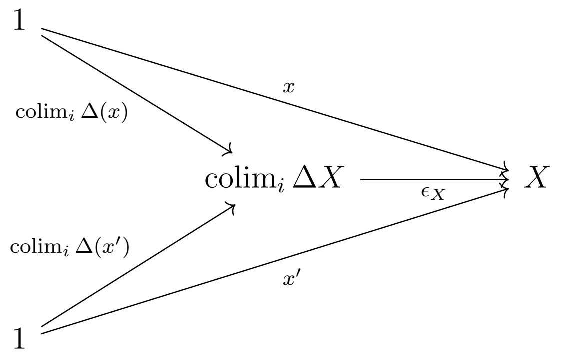
So for any set X and any $x \in X$, apply **Nat. ϵ** to $x : 1 \to X$:



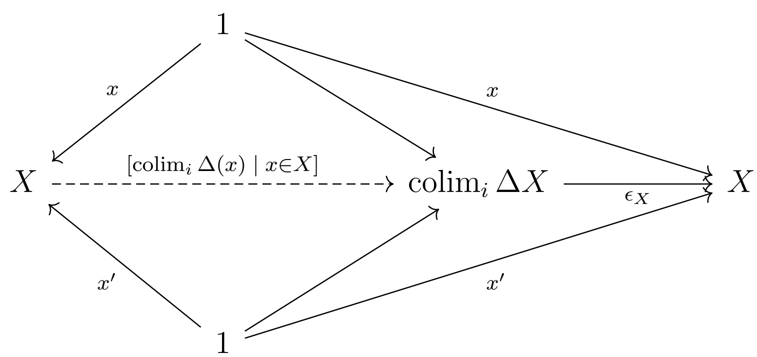
Now we'll use the fact that $\mathrm{colim}_i\, \Delta 1 = 1$ (hence ϵ_1 must be the identity on 1) to collapse this square into a triangle:



5 Now recall we had the triangle above for *every* $x \in X$, hence we have



But recall Fact 2: X is the coproduct of X-many copies of 1, where the inclusion maps are the elements of X:

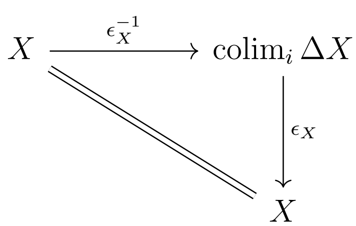


From this picture, it's quick to prove that $\epsilon_X^{-1} = [\operatorname{colim}_i \Delta(x)]$ is a right inverse for ϵ_X: for any element $x : 1 \to X$, we know that

$$x = \epsilon_X \circ \epsilon_X^{-1} \circ x$$

and thus $\epsilon_X \circ \epsilon_X^{-1} = \mathbf{id}_X$. **5**

If we could prove that ϵ_X^{-1} is a left inverse for ϵ_X, we would be done. But this proves difficult to do directly. **Instead we'll adopt a slightly different approach:** **6** we'll show that ϵ_X^{-1} is an isomorphism by other means, and then, since we just showed that $\epsilon_X \circ \epsilon_X^{-1}$ is an isomorphism (id_X), we'll have that ϵ_X is an iso by the **3-for-2 property of isomorphisms.** **7**



6 If we view **8** ϵ_X^{-1} as a morphism **9** $\coprod_x \operatorname{colim}_i 1 \to \operatorname{colim}_i \coprod_x 1$ and unfold the definition

of ϵ_X^{-1}, we get:

$$\begin{aligned}
\epsilon_X^{-1} &= \left[\operatorname*{colim}_i \Delta(x) \;\mid\; x \in X\right] \\
&= \left[\left[\mathsf{inc}_j^{\Delta X} \circ (\Delta x)_j \;\mid\; j \in I\right] \;\mid\; x \in X\right] \\
&= \left[\left[\mathsf{inc}_j^{\Delta X} \circ x \;\mid\; j \in I\right] \;\mid\; x \in X\right] && \text{(Defn of } \Delta) \\
&= \left[\left[\mathsf{inc}_j^{\Delta X} \circ \mathsf{inc}_x^{\Delta x(1)} \;\mid\; j \in I\right] \;\mid\; x \in X\right] && \text{(Defn of } \mathsf{inc}^{\Delta x(1)})
\end{aligned}$$

But on Homework 6, Exercise 4 we proved that $\coprod_x \operatorname{colim}_i 1$ is isomorphic to $\operatorname{colim}_i \coprod_x 1$, and this is exactly **the isomorphism witnessing that fact** 10. So ϵ_X^{-1} is an iso, and, as mentioned above, this gives us that ϵ_X is an iso. Since X was arbitrary, we get that ϵ is a natural isomorphism and, by Fact 1, Δ is fully faithful.

Notes:

1 I'll be needing this fact later, so might as well mention it now

2 Recall that if $z_i : X(i) \to Z$ is a cocone on a diagram X with apex Z, then $[z_i \mid i \in I]$ is the unique map $\operatorname{colim}_i X(i) \to Z$ making all the triangles commute.

3 If you know one of the categories you're working with is Set, then don't hesitate to use the features of Set to solve the problem (it might be necessary to). As far as possible, I'd encourage you to try to stick to category-theoretic properties of Set (i.e. statements like Fact 2 which are phrased in terms of mappings and universal constructions – in this case, a coproduct – rather than anything too "nitty-gritty" about sets and the $\in$ relation.

4 Though I try to be careful about when I'm using x as an element of X and when I'm viewing it as a morphism $1 \to X$, there isn't actually much of a distinction. Not only are the morphisms $1 \to X$ in canonical bijection with the elements of X, but they characterize the categorical behavior of X as an object of Set: if $f, g : X \to Y$ are functions and

$$f \circ x = g \circ x$$

for all $x : 1 \to X$, then $f = g$. This is just a categorical formulation of the principle of function extensionality.

5 I'm using here the extensionality principle outlined in 4. If I had shown that $f \circ h = g \circ h$ for some f, g, h, that would not imply $f = g$ in general. I'm allowed to conclude that here because the h I proved it for was an arbitrary element $1 \to X$, i.e. for all $x \in X$.

6 Sometimes workarounds like this will save you a lot of tedious morphism algebra. After all, I don't care *what* exactly the inverse of ϵ_X is (though it's nice to know that it's the ϵ_X^{-1} given), I just care *that* ϵ_X has an inverse. Of course, since we showed that ϵ_X^{-1} is a right inverse for ϵ_X and we'll show that ϵ_X is an iso, we could then confirm that ϵ_X^{-1} is the inverse of ϵ_X by uniqueness of inverses.

7 You proved this on the first homework. The 3-for-2 property is not unique to isomorphisms, and indeed many of the notions of equivalence we care about exhibit a 3-for-2 property. It comes in handy often, like here.

8 Since I'm viewing $\operatorname{colim}_i \Delta 1 = 1$, I can use the maps $\operatorname{colim}_i \Delta 1$ is known to have (e.g. its colimit inclusion maps) as maps into 1, and vice versa. Similarly with viewing $\coprod_x 1 = X$.

9 Notice here that I suppress reference to Δ. Writing $\operatorname{colim}_i 1$ means $\operatorname{colim}_i \Delta 1$, and $\coprod_x 1$ means $\coprod_x \Delta_X(1)$. Since Δ has constant value, it's justified to instead just write that value inside the colimit. I'm doing that here because the Δs would only serve to clutter the notation.

10 This part wasn't too explicitly part of the HW6 solutions, so let me mention it. Suppose we have a category $\mathbb{C}$ with all I-shaped and J-shaped colimits, for some small I, J. Then if we have a diagram $X : I \times J \to \mathbb{C}$, write

- $\mathsf{inc}^1_{i_0,j_0}$ for the colimit inclusion $X(i_0, j_0) \to \operatorname{colim}_i X(i, j_0)$
- $\mathsf{inc}^2_{i_0,j_0}$ for the colimit inclusion $X(i_0, j_0) \to \operatorname{colim}_j X(i_0, j)$
- inc_{i_0} for the colimit inclusion $\operatorname{colim}_j X(i_0, j) \to \operatorname{colim}_i \operatorname{colim}_j X(i, j)$
- inc_{j_0} for the colimit inclusion $\operatorname{colim}_i X(i, j_0) \to \operatorname{colim}_j \operatorname{colim}_i X(i, j)$

So then we have:

$$\Big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\big|\; i \in I\big] \;\Big|\; j \in J\Big] : \operatorname*{colim}_j \operatorname*{colim}_i X(i, j) \to \operatorname*{colim}_i \operatorname*{colim}_j X(i, j) \qquad (\star)$$

$$\Big[\big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j} \;\big|\; j \in J\big] \;\Big|\; i \in I\Big] : \operatorname*{colim}_i \operatorname*{colim}_j X(i, j) \to \operatorname*{colim}_j \operatorname*{colim}_i X(i, j) \qquad (\star\star)$$

The first of these is constructed as follows: note that

$$\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} : X(i, j) \to \operatorname*{colim}_i \operatorname*{colim}_j X(i, j)$$

These maps form a cocone over $X(-, j)$, so we can pair them up along I:

$$\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\big|\; i \in I\big] : \operatorname*{colim}_i X(i, j) \to \operatorname*{colim}_i \operatorname*{colim}_j X(i, j)$$

But these maps form a cocone over $\operatorname{colim}_i X(i, -)$, so we can pair them up along J:

$$\Big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\big|\; i \in I\big] \;\Big|\; j \in J\Big] : \operatorname*{colim}_j \operatorname*{colim}_i X(i, j) \to \operatorname*{colim}_i \operatorname*{colim}_j X(i, j).$$

Giving us the map $(\star)$. We wish to show that the maps given in $(\star)$ and $(\star\star)$ are mutually inverse, and are therefore isomorphisms.

Now, observe:

$$\begin{aligned}
&\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \circ \big[\big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\big]\big] \\
&= \Big[\big[\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \circ \big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\big]\big]\Big] && (\text{Left Compose}) \\
&= \Big[\big[\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \circ \mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\big]\Big] && (\text{Left Compose}) \\
&= \Big[\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big] \circ \mathsf{inc}^1_{i,j}\big]\Big] && (\text{Cocone Triangle}) \\
&= \big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] && (\text{Cocone Triangle}) \\
&= \Big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\big|\; j \in J\big] \;\Big|\; i \in I\Big] && (*)
\end{aligned}$$

Here, Left Compose is the fact that $f \circ [z_k \mid k \in K] = [f \circ z_k \mid k \in K]$ for any suitably-typed K, z_k, and f. The dual of this property is articulated in note **12** of the HW6 solutions. The fact denoted Cocone Triangle is that $[z_k \mid k \in K] \circ \mathsf{inc}_{k_0} = z_{k_0}$ for any fixed k_0, which is true by the definition of $[-]$ (this is the claim that the triangles commute in the colimit diagram). Now, the morphism we got in line (*) is actually the identity on $\mathrm{colim}_i \, \mathrm{colim}_j \, X(i,j)$. To see this, pick any $i_0 \in I$ and $j_0 \in J$. Then observe:

$$\begin{aligned}
&\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; j \in J\big] \;\mid\; i \in I\big] \circ \mathsf{inc}_{i_0} \circ \mathsf{inc}^2_{i_0,j_0} \\
&= \big[\mathsf{inc}_{i_0} \circ \mathsf{inc}^2_{i_0,j} \;\mid\; j \in J\big] \circ \mathsf{inc}^2_{i_0,j_0} \\
&= \mathsf{inc}_{i_0} \circ \mathsf{inc}^2_{i_0,j_0} \\
&= \mathrm{id} \circ \mathsf{inc}_{i_0} \circ \mathsf{inc}^2_{i_0,j_0}
\end{aligned}$$

Since j_0 was arbitrary, we can conclude by **ColimInj** (see HW6 sols once again) that

$$\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; j \in J\big] \;\mid\; i \in I\big] \circ \mathsf{inc}_{i_0} \quad = \quad \mathrm{id} \circ \mathsf{inc}_{i_0}$$

but i_0 was arbitrary, so, again by **ColimInj**,

$$\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} \;\mid\; j \in J\big] \;\mid\; i \in I\big] \quad = \quad \mathrm{id},$$

so

$$\big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \circ \big[\big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\big]\big] \quad = \quad \mathrm{id}.$$

A similar calculation in the other direction shows that

$$\big[\big[\mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}\big]\big] \circ \big[\big[\mathsf{inc}_i \circ \mathsf{inc}^2_{i,j}\big]\big] \quad = \quad \mathrm{id}.$$