Category Theory (80-413/713) F20 HW8, Exercises 1 & 6 Solution

Jacob Neumann, November 2020

Problem:

For an adjunction

$$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, bend left, "U"] \end{tikzcd}$$

with bijection

$$\phi : \mathsf{Hom}_{\mathbb{B}}(F(-),-) \xrightarrow{\sim} \mathsf{Hom}_{\mathbb{A}}(-,U(-))$$

and objects A of $\mathbb{A}$ and B of $\mathbb{B}$, define

$$\eta_A : A \to UFA \qquad \eta_A = \phi_{A,FA}(\mathrm{id}_{FA})$$
$$\epsilon_B : FUB \to B \qquad \epsilon_B = \phi^{-1}_{UB,B}(\mathrm{id}_{UB})$$

(a) Show that η_A and ϵ_B are natural in A and B, respectively, and thus consitute natural transformations

$$\eta : \mathrm{id}_{\mathbb{A}} \to U \circ F \qquad \epsilon : F \circ U \to \mathrm{id}_{\mathbb{B}}$$

called the *unit* and *counit* of the adjunction.

(b) Show that the triangles

$$\begin{tikzcd} UB \arrow[d, "\eta_{UB}"'] \arrow[dr, equal, "\mathrm{id}_{UB}"] & \\ UFUB \arrow[r, "U(\epsilon_B)"'] & UB \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} FA \arrow[d, "F(\eta_A)"'] \arrow[dr, equal, "\mathrm{id}_{FA}"] & \\ FUFA \arrow[r, "\epsilon_{FA}"'] & FA \end{tikzcd}$$

(called *triangle equalities*) commute for all $A \in \mathbb{A}$ and $B \in \mathbb{B}$.

Solution:

1

First, one point of notation: I'll generally omit the subscripts when I use ϕ, e.g. writing $\eta_A = \phi(\mathrm{id}_{FA})$. This is done for convenience and readability; you can also test your understanding by trying to determine which component of ϕ we're applying. If we were working with ϕ as a natural transformation more in its own right (e.g. composing it with other natural transformations), then we would need to take more care to distinguish ϕ from its components.

2

The solution to this problem entirely hinges on correctly stating the naturality of ϕ and ϕ^{-1}. While we could state what it means for ϕ, ϕ^{-1} to be natural in A and B simultaneously (i.e. **articulate Hom$(F(-),-)$ and Hom$(-,U(-))$ as functors $\mathbb{A}^{\mathrm{op}} \times \mathbb{B} \to$ Set and state ϕ and ϕ^{-1} as natural transforms between them** 1), it is more fruitful to state the naturality in A and naturality in B separately.

To state the naturality in A, we fix an arbitrary object B of $\mathbb{B}$. Then for any $f : A \to A'$ in $\mathbb{A}$, the following diagram must commute.

$$\begin{tikzcd}
\mathsf{Hom}(FA, B) \arrow[r, bend left, "\phi"] & \mathsf{Hom}(A, UB) \arrow[l, bend left, "\phi^{-1}"] \\
\mathsf{Hom}(FA', B) \arrow[u, "-\circ F(f)"] \arrow[r, bend left, "\phi"] & \mathsf{Hom}(A', UB) \arrow[l, bend left, "\phi^{-1}"] \arrow[u, "-\circ f"']
\end{tikzcd}$$

Note the contravariance of $\mathsf{Hom}(F(-), B)$ and $\mathsf{Hom}(-, UB)$: f goes from A to A', but $\mathsf{Hom}(Ff, B)$ goes from $\mathsf{Hom}(FA', B)$ to $\mathsf{Hom}(FA, B)$. As indicated in the diagram, the morphism part of the $\mathsf{Hom}(F(-), B)$ and $\mathsf{Hom}(-, UB)$ functors is given by precomposition. For our later convenience, let us state these as: **2**

Eq. 1 $\phi(v \circ F(f)) = \phi(v) \circ f$ (For all $v \in \mathsf{Hom}_{\mathbb{B}}(FA', B)$)

Eq. 2 $\phi^{-1}(u) \circ F(f) = \phi^{-1}(u \circ f)$ (For all $u \in \mathsf{Hom}_{\mathbb{A}}(A', UB)$)

Similarly, to state the naturality in B, we fix any object A of $\mathbb{A}$. Then, for any $g : B \to B'$ in $\mathbb{B}$, the following commutes.

$$\begin{tikzcd}
\mathsf{Hom}(FA, B) \arrow[r, bend left, "\phi"] \arrow[d, "g\circ-"'] & \mathsf{Hom}(A, UB) \arrow[l, bend left, "\phi^{-1}"] \arrow[d, "U(g)\circ-"] \\
\mathsf{Hom}(FA, B') \arrow[r, bend left, "\phi"] & \mathsf{Hom}(A, UB') \arrow[l, bend left, "\phi^{-1}"]
\end{tikzcd}$$

Eq. 3 $\phi(g \circ v) = U(g) \circ \phi(v)$ (For all $v \in \mathsf{Hom}_{\mathbb{B}}(FA, B)$)

Eq. 4 $g \circ \phi^{-1}(u) = \phi^{-1}(U(g) \circ u)$ (For all $u \in \mathsf{Hom}_{\mathbb{A}}(A, UB)$)

These four equations will allow us to **solve (a) and (b)** **3**.

3

For (a), let's start with the naturality of η. Pick arbitrary $f : A \to A'$ in $\mathbb{A}$. We need to argue that the **naturality square** 4

$$\begin{tikzcd} A \arrow[r, "\eta_A"] \arrow[d, "f"'] & UFA \arrow[d, "UFf"] \\ A' \arrow[r, "\eta_{A'}"'] & UFA' \end{tikzcd}$$

commutes. But, with the definition of η, Eq. 1, and Eq. 3, this becomes **a simple calculation** 5:

$$\begin{aligned} \eta_{A'} \circ f &= \phi(\mathrm{id}_{FA'}) \circ f \\ &= \phi(\mathrm{id}_{FA'} \circ F(f)) && \text{Eq. 1} \\ &= \phi(F(f) \circ \mathrm{id}_{FA}) \\ &= U(F(f)) \circ \phi(\mathrm{id}_{FA}) && \text{Eq. 3} \\ &= UFf \circ \eta_A \end{aligned}$$

4

The naturality of ϵ is similar: for arbitrary $g : B \to B'$ in $\mathbb{B}$, the naturality square

$$\begin{tikzcd} FUB \arrow[r, "\epsilon_B"] \arrow[d, "FUg"'] & B \arrow[d, "g"] \\ FUB' \arrow[r, "\epsilon_{B'}"'] & B' \end{tikzcd}$$

commutes:

$$\begin{aligned} \epsilon_{B'} \circ FUg &= \phi^{-1}(\mathrm{id}_{UB'}) \circ F(Ug) \\ &= \phi^{-1}(\mathrm{id}_{UB'} \circ Ug) && \text{Eq. 2} \\ &= \phi^{-1}(Ug \circ \mathrm{id}_{UB}) \\ &= g \circ \phi^{-1}(\mathrm{id}_{UB}) && \text{Eq. 4} \\ &= g \circ \epsilon_B \end{aligned}$$

5 Part (b) also follows **easily** [6] from these four equations. Pick arbitrary B in $\mathbb{B}$. Then,

$$\begin{aligned} U(\epsilon_B) \circ \eta_{UB} &= U(\epsilon_B) \circ \phi(\mathrm{id}_{FUB}) \\ &= \phi(\epsilon_B \circ \mathrm{id}_{FUB}) && \text{Eq. 3} \\ &= \phi(\epsilon_B) \\ &= \phi(\phi^{-1}(\mathrm{id}_{UB})) \\ &= \mathrm{id}_{UB}. \end{aligned}$$

For arbitrary A in $\mathbb{A}$,

$$\begin{aligned} \epsilon_{FA} \circ F(\eta_A) &= \phi^{-1}(\mathrm{id}_{UFA}) \circ F(\eta_A) \\ &= \phi^{-1}(\mathrm{id}_{UFA} \circ \eta_A) && \text{Eq. 2} \\ &= \phi^{-1}(\eta_A) \\ &= \phi^{-1}(\phi(\mathrm{id}_{FA})) \\ &= \mathrm{id}_{FA} \end{aligned}$$

Notes:

[1] More Formally,

$$\begin{aligned} \mathsf{Hom}_{\mathbb{B}}(F(-),-) : \mathbb{A}^{\mathrm{op}} \times \mathbb{B} &\longrightarrow \mathsf{Set} \\ (A,B) &\mapsto \mathsf{Hom}_{\mathbb{B}}(FA,B) \\ (f : A \to A', g : B \to B') &\mapsto (v \mapsto g \circ v \circ F(f)) : \mathsf{Hom}_{\mathbb{B}}(FA',B) \to \mathsf{Hom}_{\mathbb{B}}(FA,B') \end{aligned}$$

$$\begin{aligned} \mathsf{Hom}_{\mathbb{A}}(-,U(-)) : \mathbb{A}^{\mathrm{op}} \times \mathbb{B} &\longrightarrow \mathsf{Set} \\ (A,B) &\mapsto \mathsf{Hom}_{\mathbb{A}}(A,UB) \\ (f : A \to A', g : B \to B') &\mapsto (u \mapsto U(g) \circ u \circ f) : \mathsf{Hom}_{\mathbb{A}}(A',UB) \to \mathsf{Hom}_{\mathbb{A}}(A,UB') \end{aligned}$$

A morphism in $\mathbb{A}^{\mathrm{op}} \times \mathbb{B}$ is a pair $(f,g) : (A',B) \to (A,B')$ where $f : A \to A'$ in $\mathbb{A}$ and $g : B \to B'$ in $\mathbb{B}$. For ϕ to be a natural transformation $\mathsf{Hom}(F(-),-) \to \mathsf{Hom}(-,U(-))$ means that for all (f,g) in $\mathbb{A}^{\mathrm{op}} \times \mathbb{B}$, the diagram

$$\begin{tikzcd} \mathsf{Hom}_{\mathbb{B}}(FA',B) \arrow[r, bend left, "\phi"] \arrow[d, "g \circ - \circ F(f)"'] & \mathsf{Hom}_{\mathbb{A}}(A',UB) \arrow[l, bend left, "\phi^{-1}"] \arrow[d, "Ug \circ - \circ f"] \\ \mathsf{Hom}_{\mathbb{B}}(FA,B') \arrow[r, bend left, "\phi"] & \mathsf{Hom}_{\mathbb{A}}(A,UB') \arrow[l, bend left, "\phi^{-1}"] \end{tikzcd}$$

commutes. Facts Eq. 1 through Eq. 4 are the special cases where either $f = \mathrm{id}_A$ or $g = \mathrm{id}_B$.

2 As always, isolating and naming the key facts of the problem is always recommended.

3 Notice here that we've taken the abstract (and somewhat complex) definition for what it means for ϕ to be a natural transformation, and reduced it to four equations. Throughout this problem (and others), this is all we'll need the naturality of ϕ for: to manipulate algebraic equations involving F, U, ϕ, and ϕ^{-1}. Specifically, notice how all of these equations involve bringing f or g inside the argument to ϕ or ϕ^{-1} (and taking f or g "out" of the argument). Eq. 2, for instance, says that if we have ϕ^{-1} applied to something composed with $F(f)$ on the right, then we can bring the f inside the argument to ϕ^{-1}, dropping the F. In this way, complex and abstract notions like "the naturality of ϕ" become simple rules for manipulating symbols.

4 All naturality squares have the same structure: they're parametrized by morphisms in the domain category of the two functors. On the left is the first functor applied to the morphism (including its domain and codomain), on the right is the second functor applied to the same morphism, and the purported natural transformation runs between the two sides along the top and bottom of the square.

5 Unless you're invoking some more complex argument (and know what you're doing), then this is how you should usually perform a naturality calculation. Establish all the facts you need (like I've done here with Eq. 1 and Eq. 3) and then grind out the algebra so it's clear to see.

6 As much as I love seeing a bunch of diagrams and explanation, here it's not really necessary. The diagrams I'm proving have already been drawn (in the problem statement), and I've thoroughly established the keys facts I'll need. So no need to waste time – let us calculate!

Problem:

> Suppose we have an adjunction 7
>
> $$\begin{tikzcd} \mathbb{A} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbb{B} \arrow[l, bend left, "U"] \end{tikzcd}$$
>
> and a diagram $X : I \to \mathbb{A}$ such that $\mathrm{colim}_i X_i$ exists in $\mathbb{A}$.
>
> (a) Construct a bijection $\mathsf{cocone}(F \circ X, Z) \cong \mathsf{cocone}(X, U(Z))$
>
> (b) Use this to prove that $F(\mathrm{colim}_i X_i)$ is a colimit for the diagram $(F \circ X) : I \to \mathbb{B}$.

Solution:

6

Let ϕ be the natural transformation **witnessing** $F \dashv U$ 8, as above. In particular, Eq. 1 through Eq. 4 still hold.

(a) We'll construct a bijection 9

$$\Theta_Z : \mathsf{cocone}(F \circ X, Z) \xrightarrow{\sim} \mathsf{cocone}(X, U(Z)).$$

Let $(z_i : F(X_i) \to Z)_{i \in I}$ be a cocone on $F \circ X$ with apex Z in $\mathbb{B}$. In particular,

Eq. 5 $\quad z_i = z_j \circ F(X(w)) \qquad$ (For all $w : i \to j$ in I)

Then define $\Theta_Z((z_i)_{i \in I})$ by putting

$$\theta_i = \phi(z_i) : X_i \to U(Z)$$

for each object i of I. To see that this defines a cocone on X with apex $U(Z)$, pick arbitrary $w : i \to j$ in I and observe

$$\begin{aligned} \theta_j \circ X(w) &= \phi(z_j) \circ X(w) & \\ &= \phi(z_j \circ F(X(w))) & \text{Eq. 1} \\ &= \phi(z_i) & \text{Eq. 5} \\ &= \theta_j. & \end{aligned}$$

Finally, to see that Θ_Z is a bijection, it suffices to define Θ_Z^{-1} as the function sending a cocone $(\zeta_i : X_i \to U(Z))_{i \in I}$ to $(\phi^{-1}(\zeta_i) : F(X_i) \to Z)_{i \in I}$, which can be seen to be a cocone similarly. This is an inverse for Θ_Z because ϕ and ϕ^{-1} are inverses, hence for any cocones $(z_i : F(X_i) \to Z)$ and $(\zeta_i : X_i \to U(Z))$, we have for any i that

$$\phi(\phi^{-1}(\zeta_i)) = \zeta_i \qquad \text{and} \qquad \phi^{-1}(\phi(z_i)) = z_i$$

so Θ_Z and Θ_Z^{-1} are inverses. 10

7

For (b), let

$$\mathsf{inc}_j : X_j \to \operatorname*{colim}_i X_i \qquad (j \in I)$$

denote the components of the colimit cocone on X. We need to show that

$$\left(F(\mathsf{inc}_j) : F(X_j) \to F(\operatorname*{colim}_i X_i) \right)_{j \in I} \quad \text{is a colimit cocone on } F \circ X.$$

A quick argument goes like this: take any object Z of $\mathbb{B}$, and observe the following bijections

$$\begin{aligned} \mathsf{Hom}(F(\operatorname*{colim}_i X_i), Z) &\cong \mathsf{Hom}(\operatorname*{colim}_i X_i, U(Z)) && (F \dashv U) \\ &\cong \mathsf{cocone}(X, U(Z)) && \text{(Univ. Prop. of Colimits)} \\ &\cong \mathsf{cocone}(F \circ X, Z). && (\Theta, \Theta^{-1}) \end{aligned}$$

We can check 11 that for any $k \in \mathsf{Hom}(F(\operatorname{colim}_i X_i), Z)$ the cocone on $F \circ X$ we get out of this bijection is indeed the one given by $k \circ F(\mathsf{inc}_i) : F(X_i) \to Z$, thus we satisfy the **universal property of colimits 12**.

8

We'll work through this solution in more elementary detail. As usual, proving that a cocone is a colimit requires us to verify the universal property of colimits. Pick an arbitrary cocone

$$m_i : F(X_i) \longrightarrow M.$$

We can apply Θ_M to this cocone, and obtain a cocone

$$\phi(m_i) : X_i \longrightarrow U(M).$$

But by the universal property of $\operatorname{colim}_i X_i$, we obtain a unique map $h : \operatorname{colim}_i X_i \to U(M)$ such that

Eq. 6 $\quad \phi(m_i) = h \circ \mathsf{inc}_i \qquad$ (For all $i \in I$)

We claim that $\phi^{-1}(h) : F(\operatorname{colim}_i X_i) \to M$ fulfills the universal property of colimits, namely that $\phi^{-1}(h)$ is the unique map $k : F(\operatorname{colim}_i X_i) \to M$ such that

$$m_i = k \circ F(\mathsf{inc}_i) \qquad \text{for all } i \in I.$$

To see that $\phi^{-1}(h)$ is such a k, observe that

$$\begin{aligned} m_i &= \phi^{-1}(\phi(m_i)) \\ &= \phi^{-1}(h \circ \mathsf{inc}_i) && \text{Eq. 6} \\ &= \phi^{-1}(h) \circ F(\mathsf{inc}_i) && \text{Eq. 2} \end{aligned}$$

To see that it is the *unique* such k, suppose we had another $k' : F(\operatorname{colim}_i X_i) \to M$ such that

Eq. 7 $m_i = k' \circ F(\mathsf{inc}_i)$ (For all i)

Then consider $\phi(k') : \operatorname{colim}_i X_i \to U(M)$, and observe

$$\begin{aligned} \phi(k') \circ \mathsf{inc}_i &= \phi(k' \circ F(\mathsf{inc}_i)) && \text{Eq. 1} \\ &= \phi(m_i) && \text{Eq. 7} \end{aligned}$$

But remember that we said h was the *unique* map $\operatorname{colim}_i X_i \to U(M)$ such that $h \circ \mathsf{inc}_i = \phi(m_i)$ for all i. Thus we get

$$\phi(k') = h$$

and, by the bijectivity of ϕ,

$$k' = \phi^{-1}(h)$$

as desired.

Notes:

7 I've changed the names of the categories and functors of this problem to match that of the previous one. You're welcome to change the variable names & such from the problem statement when writing your solutions, so long as you make it super obvious you're still solving the same problem.

8 I very much like this terminology of "witnesses": we're saying that $F \dashv U$, and that ϕ is the piece of data (satisfying the requisite properties) that "fulfills" the requirements put forth in the definition of adjunction, hence it *bears witness* to the fact that $F \dashv U$.

9 Note that I indexed Θ with the apex of the cocone, Z. This is because there is such a Θ for each object Z of $\mathbb{B}$. In the problem, I'll be using different instances of Θ.

10 I'm implicitly using the fact that cocones are defined by their cocone maps. So if $(d_i : X_i \to C)_{i \in I}$ and $(e_i : X_i \to C)_{i \in I}$ are two cocones on the same diagram with the same apex such that

$$e_i = d_i \qquad \text{for all } i \in I,$$

then $(d_i)_{i \in I} = (e_i)_{i \in I}$, i.e. they are equal elements of $\mathsf{cocone}(X, C)$.

11 Label this chain of bijections as

$$\begin{aligned} \mathsf{Hom}(F(\operatorname*{colim}_i X_i), Z) &\xrightarrow{\sim} \mathsf{Hom}(\operatorname*{colim}_i X_i, U(Z)) && (1) \\ &\xrightarrow{\sim} \mathsf{cocone}(X, U(Z)) && (2) \\ &\xrightarrow{\sim} \mathsf{cocone}(F \circ X, Z). && (3) \end{aligned}$$

Starting with $k \in \mathsf{Hom}(F(\operatorname{colim}_i X_i), Z)$ on the left, we have

$$k \overset{(1)}{\mapsto} \phi(k) \overset{(2)}{\mapsto} (\phi(k) \circ \mathsf{inc}_i)_{i \in I} \overset{(3)}{\mapsto} \Theta^{-1}((\phi(k) \circ \mathsf{inc}_i)_{i \in I})$$

But Θ^{-1} applied to the cocone has i-component $\phi^{-1}(\phi(k) \circ \mathsf{inc}_i)$. Using **Eq. 2** gives

$$\phi^{-1}(\phi(k) \circ \mathsf{inc}_i) = \phi^{-1}(\phi(k)) \circ F(\mathsf{inc}_i) = k \circ F(\mathsf{inc}_i) \qquad \text{for all } i \in I$$

as claimed. This fulfills the universal property of colimits because we know that each of these steps is a bijection, i.e. for any cocone on $F \circ X$ with apex Z, we can recover a unique map in $\mathsf{Hom}(F(\operatorname{colim}_i X_i), Z)$ which, when run through (1),(2),(3) in order, gives us back the cocone we started with, i.e. it makes all the triangles commute. See **12**.

12 Remember that, for any J-shaped diagram Y, the universal property of colimits is not merely that *there is* a bijection $\mathsf{Hom}(\operatorname{colim}_j Y_j, Z) \cong \mathsf{cocone}(Y, Z)$ for every Z, but specifically that this bijection is the function $k \mapsto (k \circ \mathsf{inc}_j)_{j \in J}$. So it was not enough to just demonstrate the bijections – we needed to do the work in **11** to show that the bijection we obtained was in fact this canonical function.