Category Theory (80-413/713) F20 HW7, Exercise 5 Solution

Jacob Neumann, October 2020

Problem:

Consider the category of diagrams $\mathsf{Set}^{\to} = \mathsf{Fun}([1], \mathsf{Set})$. Explicitly, the objects are maps $f : A_0 \to A_1$ in Set and the morphisms $(f : A_0 \to A_1) \to (g : B_0 \to B_1)$ are pairs $(u, v) \in \mathsf{Hom}(A_0, B_0) \times \mathsf{Hom}(A_1, B_1)$ such that the square commutes

$$\begin{array}{ccc} A_0 & \xrightarrow{u} & B_0 \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B_1 & \xrightarrow[v]{} & B_1 \end{array}$$

We consider the constant diagram functor

$$\Delta : \mathsf{Set} \to \mathsf{Set}^{\to}$$

given by $\Delta(E) = 1_E : E \to E$, $\Delta(f : A_0 \to A_1) = (f, f) : 1_{A_0} \to 1_{A_1}$ and the two evaluation functors $\mathsf{ev}_0, \mathsf{ev}_1 : \mathsf{Set}^{\to} \to \mathsf{Set}$ given by

$$\mathsf{ev}_0(f : A_0 \to A_1) = A_0 \qquad \mathsf{ev}_1(f : A_0 \to A_1) = A_1$$

(a) Prove that $\mathsf{ev}_1 \dashv \Delta$

(b) Prove that $\Delta \dashv \mathsf{ev}_0$

(c) Find a left adjoint to ev_1

(d) Find a right adjoint to ev_0

Solution:

1

(a) We must show for all functions $f : A_0 \to A_1$ and all sets B that

$$\mathbf{Hom}(\mathbf{ev}_1(f), B) \cong \mathbf{Hom}_{\to}(f, \Delta(B))$$

which, applying the definitions of these functors, [1] is the same as

$$\mathsf{Hom}(A_1, B) \cong \mathsf{Hom}_{\to}(f, 1_B).$$

So we must construct a $\varphi_{f,B}$ taking each $g : A_1 \to B$ to a commutative square with f and 1_B as opposite sides. Since 1_B is the identity, **this collapses to a triangle.** [2]

$$\varphi_{f,B} : \quad \left(A_1 \xrightarrow{g} B \right) \quad \mapsto \quad \begin{array}{ccc} A_0 & & \\ {\scriptstyle f}\downarrow & \searrow^{?} & \\ A_1 & \xrightarrow[?]{} & B \end{array}$$

As the previous diagram suggests, we should pick g and $g \circ f$ to fill the two question marks, i.e. define

$$\varphi_{f,B}(g) = (g \circ f, g)$$

One can check that this indeed produces a morphism in $\mathsf{Set}^{\to}$ from f to 1_B, because $g \circ f = 1_B \circ (g \circ f)$. **3** To get that φ is bijective, we supply its inverse:

$$\varphi^{-1}_{f,B}(u, v) = v.$$

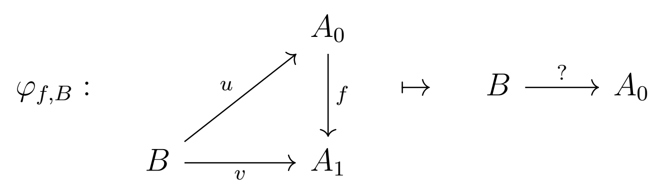
A quick calculation will show that $\varphi^{-1}_{f,B} \circ \varphi_{f,B}$ and $\varphi_{f,B} \circ \varphi^{-1}_{f,B}$ are the respective identity functions. Demonstrating the latter requires that we remember that $(u, v) : f \to 1_B$ entails that $u = v \circ f$. **4**

2

(b) For all $f : A_0 \to A_1$ and all B, we want to show

$$\mathsf{Hom}_{\to}(1_B, f) \cong \mathsf{Hom}(B, A_0)$$

which requires a bijection $\varphi_{f,B} : \mathsf{Hom}_{\to}(1_B, f) \to \mathsf{Hom}(B, A_0)$:



As the diagram suggests, we put

$$\varphi_{f,B}(u, v) = u.$$

which **clearly** **5** produces a morphism of the correct type. Its inverse is then given by

$$\varphi^{-1}_{f,B}(g) = (g, f \circ g).$$

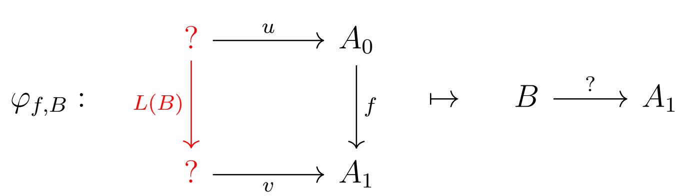
We can verify this is indeed an inverse for $\varphi_{f,B}$ by an application of their definitions and the fact that $(u, v) \in \mathsf{Hom}_{\to}(1_B, f)$ implies $v = f \circ u$. **6**

3

(c) We'll call the left adjoint $L : \mathsf{Set} \to \mathsf{Set}^{\to}$. We want to define L such that, for all $f : A_0 \to A_1$ and all B,

$$\mathsf{Hom}_{\to}(L(B), f) \cong \mathsf{Hom}(B, A_1).$$

Like before, we depict this as:



The red portion (including the red question marks) **7** is what we must define as part of our definition of L, and the other unknown (the map $B \to A_1$ on the right) is given by our definition of $\varphi_{f,B}$. We want to define L in such a way that all the **"information"** [8] in the square on the left is given by just one morphism $B \to A_1$, so we can define $\varphi_{f,B}^{-1}$ which uniquely "recovers" the square from just the single morphism.

4

We'll start with determining the codomain of $L(B)$. We're interested in squares which are determined by a map $B \to A_1$, so a **plausible choice** [9] is to define $L(B)$ to have codomain B. That gives us this diagram

$$\varphi_{f,B} : \quad \begin{array}{ccc} ? & \xrightarrow{u} & A_0 \\ {\scriptstyle L(B)}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[v]{} & A_1 \end{array} \quad \mapsto \quad B \xrightarrow{?} A_1$$

which immediately suggests the definition

$$\varphi_{f,B}(u, v) = v$$

i.e. $\varphi_{f,B}$ maps each square to its bottom map. If we use this definition, then we need to pick $\mathrm{dom}(L(B))$ and $L(B)$ such that for every $v : B \to A_1$ there exists a unique map

$$u : \mathrm{dom}(L(B)) \to A_0$$

such that $v \circ L(B) = f \circ u$. This will make it possible to define an inverse to $\varphi_{f,B}$. **It turns out that picking $\mathrm{dom}(L(B)) = \emptyset$ will work for exactly this purpose.** [10]

$$\varphi_{f,B} : \quad \begin{array}{ccc} \emptyset & \xrightarrow{u} & A_0 \\ {\scriptstyle 0_B}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[v]{} & A_1 \end{array} \quad \mapsto \quad B \xrightarrow{?} A_1$$

Here, for any set X, I write 0_X for the unique function $\emptyset \to X$. So our candidate definition of $L(B)$ is $0_B : \emptyset \to B$, and our candidate $\varphi_{f,B}$ takes (u, v) to v. We must now check that this works.

5

To check the functoriality of L, we first need to supply a morphism part. So for any $h : X \to Y$ in Set, we need an $L(h) = (u_h, v_h)$ such that

$$\begin{array}{ccc} \emptyset & \xrightarrow{u_h} & \emptyset \\ {\scriptstyle 0_X}\downarrow & & \downarrow{\scriptstyle 0_Y} \\ X & \xrightarrow[v_h]{} & Y \end{array}$$

commutes. The only available choice is $v_h = h$ and $u_h = 1_\emptyset$, and this commutes trivially. The functoriality conditions follow quickly, by definition. [11]

Since we do not need to prove $\varphi_{f,B}$ natural, it suffices to exhibit an inverse of $\varphi_{f,B}$.

Given $g : B \to A_1$, we have the following.

$$\begin{array}{ccc} \emptyset & & A_0 \\ {\scriptstyle L(B)}\downarrow & & \downarrow{\scriptstyle f} \\ B & \xrightarrow[g]{} & A_1 \end{array}$$

Now, 0_{A_0} is the unique map which can go on the top of this square to make it commute, since 0_{A_0} is the unique map $\emptyset \to A_0$ (and 0_{A_0} does indeed make the square commute). Therefore, it suffices to put

$$\varphi^{-1}_{f,B}(g) = (0_{A_0}, g).$$

We can see that $\varphi^{-1}_{f,B}$ is indeed an inverse for $\varphi_{f,B}$ using the fact that, for each g, 0_{A_0} is the unique map $\emptyset \to A_0$ making the above square commute. So we are done.

6

(d) The process for solving this is similar to (indeed, dual to) that of solving (c), so we'll just present the details of the solution. We'll call the desired right adjoint $R : \mathsf{Set} \to \mathsf{Set}^{\to}$. For any set B, define $R(B)$ to be the unique map

$$B \xrightarrow{R(B)=!_B} 1$$

where 1 is the terminal object of Set. It's easy to define the morphism part and check functoriality, analogous to the case for (c).

Now, we want

$$\varphi_{f,B} : \mathsf{Hom}(A_0, B) \cong \mathsf{Hom}_{\to}(f, R(B))$$

i.e. given any $g : A_0 \to B$, we want to define u, v such that $v \circ f = R(B) \circ u$:

$$A_0 \xrightarrow{g} B \quad \mapsto \quad \begin{array}{ccc} A_0 & \xrightarrow{u} & B \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle R(B)} \\ A_1 & \xrightarrow[v]{} & 1 \end{array}$$

The obvious choice here is $u = g$ and $v =!_{A_1}$, which we can check forms a commutative square. This is indeed a bijective function:

$$\varphi_{f,B}(g) = (g, !_{A_1}) \qquad \varphi^{-1}_{f,B}(u, v) = u.$$

The fact that $\varphi_{f,B}(\varphi^{-1}_{f,B}(u, v)) = (u, v)$ for any u, v making the right square commute follows from the fact that there is a unique morphism $A_1 \to 1$, namely $!_{A_1}$. **12**

Notes:

1 It's nice when you do this: write out the definition you're applying (in this case, the bijection definition of what $\mathsf{ev}_1 \dashv \Delta$ means) *without simplifying at all*, and then simplify it into a useful form. This communicates that you know the definition and understand how to apply it. Also, if you correctly reproduce the definition but subsequently make a mistake in the simplification, then I'm still able to give you some points! If you just jump to the simplified expression and do so incorrectly, then I don't have any evidence that you understand the definition at all.

2 It would be essentially the same deal if I left these as squares with 1_B on one side instead of "collapsing" them into triangles. However, it's generally good practice to simplify your commutative diagrams in whatever way possible: a good diagram communicates one thing very clearly, and excludes any distracting elements which aren't relevant to that point. The 1_B didn't – in my opinion – contribute anything to the point I'm trying to make with the diagram.

3 There's nothing profound about this equation. But it's worth saying it, because it's the condition that must hold for my claim (that $\varphi_{f,B}(g)$ is a morphism from f to 1_B in $\mathsf{Set}^{\to}$) to be true.

4 This is – in my view – the best way to dispense with tedious calculations. Actually showing that $\varphi^{-1}_{f,B} \circ \varphi_{f,B}$ and $\varphi_{f,B} \circ \varphi^{-1}_{f,B}$ would be boring (to read and write), and contains almost no interesting insight into the problem. So instead of doing that, I thought through the calculations in my head to see if there's anything worth mentioning. The $\varphi^{-1}_{f,B} \circ \varphi_{f,B} = 1_{\mathsf{Hom}(A_1,B)}$ calculation is so obvious I don't need to say anything more. The other direction is almost as trivial: there's exactly one step which requires any kind of insight. That step is realizing that (u, v) and $(v \circ f, v)$ are indeed the same. So what I do is supply the key insight behind performing that step. This allows me to "have my cake and eat it too": I don't have to perform annoying calculations, but no skeptical reader could accuse me of leaving out critical details.

5 It's easy to overuse "clearly", "obviously", etc. I feel justified in doing so here because I explicitly listed the relevant definition right there, and right above that is a diagram which very visually makes that point. Any time you want to use dismissive words like "clearly", I invite you to ask yourself, "where?", as in, "where on the page (or in the definitions) is the information which renders this point trivial/clear/obvious/simple/self-evident/easy to check?". If you can't answer that question, then maybe it's not so clear after all.

6 See **4**

7 Your ultimate goal is to get information into your reader's brain. Color can be a powerful way to do that if you're smart about it. But be careful: keep in mind that your reader may not see the color (e.g. they printed your proof out in black-and-white), and your proof should still make sense. I'm a little borderline here (since I refer to the "red portion"), but I'm really only using the color to highlight – there's no vital information which is *only* communicated by the color.

8 Intuitive descriptions like this can be very effective, *but only if backed up by actual formal detail.* What I'm ultimately getting at here is the fact that each map $g : B \to A_1$ should – if I define $L(B)$ right – determine a unique commutative square with $L(B)$ as its left side and f as its right. Formally, I will do this by explicitly defining $L(B)$ and exhibiting a bijection between $\mathsf{Hom}(B, A_1)$ and $\mathsf{Hom}_{\to}(L(B), f)$. But to keep my reader on board as I get there, it helps to engage their intuitions about what it means for g to uniquely determine a $(u, v) : L(B) \to f$. My reader likely has some sense of what it means to recover a larger set of information from a smaller yet determinative data set. This is what we're doing in a vague sense, and putting words in quotation marks (like I've done with "information" and "recover") communicates to the reader that these are vague – but perhaps helpful – indications of how to think. Remember: your reader is ultimately just a regular person who lives their life doing everyday stuff; if you want to shepherd them through the (often spooky and metaphysical) world of category theory, reassure them that it's not *that* different than stuff they're more used to.

9 "It's a plausible choice to define it this way" is my way of saying "I know this is the right answer, just go with it"

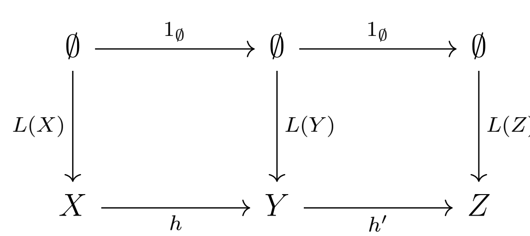
10 This is the critical insight to solving this problem. Thinking, "maybe I should try the initial object here" is part of the instinct you build up by doing lots of category theory problems. So if it wasn't obvious to you to proceed in that way, then that just means you're still learning! What led me to try $\emptyset$ as $\text{dom}(L(B))$ was two observations.

1. I need to pick $\text{dom}(L(B))$ such that there's *always* a map $\text{dom}(L(B)) \to \text{dom}(f)$ for *any* f, *any* B, and *any* $g : B \to \text{cod}(f)$. That's a lot of "any"s – I better pick a $\text{dom}(L(B))$ which will readily map into whatever $\text{dom}(f)$ might happen to be.
2. Also, I don't just need any old map $u : \text{dom}(L(B)) \to \text{dom}(f)$, I need there to be *a unique such map* for each $g : B \to \text{cod}(f)$ which makes the square commute. So basically I have a universal property that I want u to satisfy. The easiest way to prove a universal property (in general) is to use *other* universal properties. So maybe I want to pick $\text{dom}(L(B))$ such that it has a universal property I can use. What object has a universal property about uniquely mapping into a bunch of other objects?

11 The identity condition says that $L(1_B) = 1_{L(B)}$, where recall that $1_{L(B)}$ is

$$(1_{\text{dom}(L(B))}, 1_{\text{cod}(L(B))}) : L(B) \to L(B)$$

which is equal to $L(1_B)$ by the definition of the morphism part given above. For composition, if $h : X \to Y$ and $h' : Y \to Z$, then the first component (the top of the square) of $L(h)$, $L(h')$, and $L(h' \circ h)$ is $1_\emptyset$ by definition. The second component (the bottom of the square) of $L(h)$ is h, of $L(h')$ is h', and of $L(h' \circ h)$ is $h' \circ h$. So clearly we get $L(h' \circ h) = L(h') \circ L(h)$. This can be seen visually as "gluing" the squares together:



12 Again, the critical step here is that (u, v) must be the same as $(u, !_{A_1})$ if $(u, v) : f \to R(B)$. I supply the information needed to figure that out, and leave the uninteresting parts of verifying these are inverses to a skeptical reader who cares enough to write it out.