Category Theory (80-413/713) F20 HW6, Exercises 4 & 5 Solution

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Problem:

> Let I and J be two small categories and $\mathbf{C}$ a category with colimits of shape I and J. We fix a diagram
> $$X : I \times J \to \mathbf{C}$$
>
> (a) Using the universal property of colimits, define diagrams
> $$X^1 : J \to \mathbf{C} \qquad\qquad X^2 : I \to \mathbf{C}$$
> $$: j \mapsto \operatorname*{colim}_i X(i,j) \qquad\qquad : i \mapsto \operatorname*{colim}_j X(i,j)$$
>
> (b) If $Z := \operatorname{colim}_j X^1(j)$, construct a cocone $X(i,j) \to Z$
>
> (c) Prove that this cocone $X(i,j) \to Z$ is a colimit cocone.
>
> (d) Prove that $\operatorname{colim}_j X^1(j) = \operatorname{colim}_i X^2(i) = \operatorname{colim}_{i,j} X(i,j)$

Solution:

1

Let's begin with some notation. Throughout, I'll use $i, i', i'', i_0, i_1, i_2$,etc. to denote objects of I, and likewise $j, j', j'', j_0, j_1, j_2$ for objects of J. f, f_1, f_2 will denote morphisms of I and g, g_1, g_2 morphisms of J. I'll write inc to denote the inclusion maps of a diagram into its colimit, with appropriate decoration to disambiguate which diagram I'm referring to:

$$\mathsf{inc}^1_{i,j} : X(i,j) \to X^1(j) \qquad \mathsf{inc}_j : X^1(j) \to Z \qquad \mathsf{inc}_i : X^2(i) \to Z$$

Given any diagram $Y : K \to \mathbf{C}$ such that $\operatorname{colim}_k Y(k)$ exists, and a cocone $w_k : Y(k) \to W$, write

$$[w_k \mid k \in K] : \operatorname*{colim}_k Y(k) \to W$$

for the unique map h such that $w_k = h \circ \mathsf{inc}^Y_k$ for all k (the map whose existence is required by the universal property of colimits). 1

2

Throughout, **we'll make use of the following principle** 2 : if Y, K are as in the previous paragraph, and $p, q : \operatorname{colim}_k Y(k) \to E$ are some morphisms in $\mathbf{C}$ such that

$$p \circ \mathsf{inc}^Y_k = q \circ \mathsf{inc}^Y_k \text{ for all objects } k \text{ of } K$$

then we can conclude that $p = q$. We'll refer to this as **ColimInj**, since it corresponds to the injectivity of the map $\mathsf{Hom}(\operatorname{colim}_k Y(k), W) \to \mathsf{Cocone}(Y, W)$ in the bijection characterization of the universal property of colimits.

3

Let's begin with (a). First begin by noting that, for any fixed object j of J, we have the functor **3**

$$X(-, j) : I \to \mathbf{C}$$

Since $X(-, j)$ is an I-shaped diagram in $\mathbf{C}$ and $\mathbf{C}$ has all such colimits, we can take the colimit over i, for each j.

$$\begin{tikzcd}
 & & & & X(i_1, j_3) \arrow[dd, "X(f_1,j_3)"'] \arrow[dr, "\mathsf{inc}^1_{i_1,j_3}"] & \\
 & & X(i_1, j_2) \arrow[urr, "X(i_1,g_2)"] \arrow[dd, "X(f_1,j_2)"'] \arrow[dr, "\mathsf{inc}^1_{i_1,j_2}"] & & & \operatorname{colim}_i X(i, j_3) \\
X(i_1, j_1) \arrow[urr, "X(i_1,g_1)"] \arrow[dd, "X(f_1,j_1)"'] \arrow[dr, "\mathsf{inc}^1_{i_1,j_1}"] & & & \operatorname{colim}_i X(i, j_2) & X(i_2, j_3) \arrow[ur, "\mathsf{inc}^1_{i_2,j_3}"] & \\
 & \operatorname{colim}_i X(i, j_1) & X(i_2, j_2) \arrow[ur, "\mathsf{inc}^1_{i_2,j_2}"] \arrow[urr, "X(i_2,g_2)"'] & & & \\
X(i_2, j_1) \arrow[ur, "\mathsf{inc}^1_{i_2,j_1}"] \arrow[urr, "X(i_2,g_1)"'] & & & & &
\end{tikzcd}$$

As indicated in the problem statement, this assignment of objects j to the respective colimit $\operatorname{colim}_i X(i, j)$ constitutes the object part of a functor $X^1 : J \to \mathbf{C}$. So, to fulfill the definition of functor, **we must supply the morphism part, and then check functoriality 4**.

4

Supplying the morphism part of this functor means that for each morphism $g : j \to j'$ of J, we supply some $\mathbf{C}$-morphism $X^1(g) : X^1(j) \to X^1(j')$.

$$\begin{tikzcd}
 & & & & X(i_1,j_3) \arrow[dr, "\mathsf{inc}^1_{i_1,j_3}"] \arrow[dd, "X(f_1,j_3)"'] & \\
 & & X(i_1,j_2) \arrow[urr, "X(i_1,g_2)"] \arrow[dr, "\mathsf{inc}^1_{i_1,j_2}"] \arrow[dd, "X(f_1,j_2)"'] & & & \operatorname{colim}_i X(i,j_3) \\
X(i_1,j_1) \arrow[urr, "X(i_1,g_1)"] \arrow[dr, "\mathsf{inc}^1_{i_1,j_1}"] \arrow[dd, "X(f_1,j_1)"'] & & & \operatorname{colim}_i X(i,j_2) \arrow[urr, dashed] & X(i_2,j_3) \arrow[ur, "\mathsf{inc}^1_{i_2,j_3}"] & \\
 & \operatorname{colim}_i X(i,j_1) \arrow[urr, dashed] & X(i_2,j_2) \arrow[ur, "\mathsf{inc}^1_{i_2,j_2}"] \arrow[urr, "X(i_2,g_2)"'] & & & \\
X(i_2,j_1) \arrow[ur, "\mathsf{inc}^1_{i_2,j_1}"] \arrow[urr, "X(i_2,g_1)"'] & & & & &
\end{tikzcd}$$

So suppose $g_1 : j_1 \to j_2$ is some such morphism of J. We'll construct a cocone on the diagram $X(-,j_1)$ whose apex is $X^1(j_2)$, and thereby obtain a morphism $X^1(j_1) \to X^1(j_2)$ by the universal property of $X^1(j_1)$ as the colimit of $X(-,j_1)$. Consider the following composition

$$X(i,j_1) \xrightarrow{X(i,g_1)} X(i,j_2) \xrightarrow{\mathsf{inc}^1_{i,j_2}} X^1(j_2)$$

where i varies over all objects of I. **To see that these maps constitute a cocone on $X(-,j_1)$** 5, pick an arbitrary $f_1 : i_1 \to i_2$ in I and observe

$$\begin{aligned}
&(\mathsf{inc}^1_{i_2,j_2} \circ X(i_2,g_1)) \circ X(f_1,j_1) \\
&= \mathsf{inc}^1_{i_2,j_2} \circ X(f_1,j_2) \circ X(i_1,g_1) && \text{(Functoriality of } X\text{)} \\
&= \mathsf{inc}^1_{i_1,j_2} \circ X(i_1,g_1). && (\mathsf{inc}^1_{i,j_2} \text{ is a cocone on } X(-,j_2))
\end{aligned}$$

Therefore this is a cocone, so we obtain a unique map $X^1(g_1) : X^1(j_1) \to X^1(j_2)$, namely $X^1(g_1) = \left[\mathsf{inc}^1_{i,j_2} \circ X(i,g_1) \;\middle|\; i \in I\right]$ via the universal property of $X^1(j_1)$ as the colimit of $X(-,j_1)$. In particular, note that the commutativity of the triangles in the universal property says that for any $i \in I$,

$$X^1(g_1) \circ \mathsf{inc}^1_{i,j_1} = \mathsf{inc}^1_{i,j_2} \circ X(i,g_1). \qquad \textbf{(FACT 1)} \; 6$$

5

To check that X^1 preserves identities, it's sufficient (by **ColimInj**) to show that

$$X^1(1_j) \circ \mathsf{inc}^1_{i,j} = 1_{X^1(j)} \circ \mathsf{inc}^1_{i,j}$$

but this is an immediate consequence of **FACT 1**:

$$\begin{aligned} X^1(1_j) \circ \mathsf{inc}^1_{i,j} &= \mathsf{inc}^1_{i,j} \circ X(i, 1_j) && \textbf{(FACT 1)} \\ &= \mathsf{inc}^1_{i,j} \circ 1_{X(i,j)} && \text{(Functoriality of } X\text{)} \\ &= \mathsf{inc}^1_{i,j} \\ &= 1_{X^1(j)} \circ \mathsf{inc}^1_{i,j} \end{aligned}$$

To see that X^1 preserves composition, pick arbitrary $g_1 : j_1 \to j_2$ and $g_2 : j_2 \to j_3$ in J and notice

$$\begin{aligned} & X^1(g_2) \circ X^1(g_1) \circ \mathsf{inc}^1_{i,j_1} \\ &= X^1(g_2) \circ \mathsf{inc}^1_{i,j_2} \circ X(i, g_1) && \textbf{(FACT 1)} \\ &= \mathsf{inc}^1_{i,j_3} \circ X(i, g_2) \circ X(i, g_1) && \textbf{(FACT 1)} \\ &= \mathsf{inc}^1_{i,j_3} \circ X(i, g_2 \circ g_1) && \text{(Functoriality of } X\text{)} \\ &= X^1(g_2 \circ g_1) \circ \mathsf{inc}^1_{i,j_1}. && \textbf{(FACT 1)} \end{aligned}$$

Again by **ColimInj**, conclude $X^1(g_2) \circ X^1(g_1) = X^1(g_2 \circ g_1)$.

6

We can conduct the analogous construction for X^2 **7**: since $X(i_0, -)$ is a J-shaped diagram for each i_0, we can take the colimit over all js to define the object part $X^2(i) = \operatorname{colim}_j X(i, j)$. Then, for any $f_1 : i_1 \to i_2$ in I, we can check that

$$\mathsf{inc}^2_{i_2,j} \circ X(f_1, j) : X(i_1, j) \to X^2(i_2)$$

constitutes a cocone on $X(i_1, -)$, and therefore the unique map

$$\left[\mathsf{inc}^2_{i_2,j} \circ X(f_1, j) \;\middle|\; j \in J\right] : X^2(i_1) \to X^2(i_2)$$

works as $X^2(f_1)$. We get a corresponding fact:

$$X^2(f_1) \circ \mathsf{inc}^2_{i_1,j} = \mathsf{inc}^2_{i_2,j} \circ X(f_1, j) \qquad \textbf{(FACT 2)}$$

which can be used to prove the functoriality of X^2 in the same manner we did for X^1.

7

Moving on to (b), we observe that X^1 is a J-shaped diagram in $\mathbf{C}$, hence we may take its colimit. Call the colimit Z, and write $\mathsf{inc}_j : X^1(j) \to Z$ for the inclusion maps.

$$\begin{tikzcd}
 & & & & X(i_1,j_3) \arrow[dd] \arrow[dr, "\mathsf{inc}^1_{i_1,j_3}"] & \\
 & & X(i_1,j_2) \arrow[urr, "X(i_1,q)"] \arrow[dd] \arrow[dr, "\mathsf{inc}^1_{i_1,j_2}"] & & & X^1(j_3) \arrow[ddd, dashed, red, "\mathsf{inc}_{j_3}"] \\
X(i_1,j_1) \arrow[urr, "X(i_1,g_1)"] \arrow[dd, "X(f_1,j_1)"'] \arrow[dr, "\mathsf{inc}^1_{i_1,j_1}"] & & & X^1(j_2) \arrow[urr, bend left, "X^1(g_2)"] \arrow[ddrr, dashed, red, "\mathsf{inc}_{j_2}"] & X(i_2,j_3) \arrow[ur] & \\
 & X^1(j_1) \arrow[urr, bend left, "X^1(g_1)"] \arrow[drrrr, dashed, red, bend right, "\mathsf{inc}_{j_1}"] & X(i_2,j_2) \arrow[ur] \arrow[urr, "X(i_2,g_2)"'] & & & \\
X(i_2,j_1) \arrow[ur, "\mathsf{inc}^1_{i_2,j_1}"] \arrow[urr, "X(i_2,g_1)"'] & & & & & \color{red}{Z}
\end{tikzcd}$$

We must show that Z is the apex of some cocone on the diagram $X : I \times J \to \mathbf{C}$, i.e. we must supply maps $z_{i,j} : X(i,j) \to Z$ for each i, j such that $z_{i,j} = z_{i',j'} \circ X(f,g)$ for every morphism $(f,g) : (i,j) \to (i',j')$ in $I \times J$. As suggested by the diagram above, this is achieved by the composition

$$X(i,j) \xrightarrow{\mathsf{inc}^1_{i,j}} X^1(j) \xrightarrow{\mathsf{inc}_j} Z$$

So put $z_{i,j} = \mathsf{inc}_j \circ \mathsf{inc}^1_{i,j}$. To see this is a cocone, pick arbitrary $f_1 : i_1 \to i_2$ and $g_1 : j_1 \to j_2$.

$$\begin{tikzcd}
X(i_1,j_1) \arrow[drr, "\mathsf{inc}^1_{i_1,j_1}"] \arrow[ddr, "X(f_1,j_1)"] \arrow[ddd, "X(i_1,g_1)"'] & & & \\
& & X^1(j_1) \arrow[dr, "\mathsf{inc}_{j_1}"] \arrow[ddd, "X^1(g_1)"] & \\
& X(i_2,j_1) \arrow[ur, "\mathsf{inc}^1_{i_2,j_1}"'] \arrow[ddd, "X(i_2,g_1)"'] & & Z \\
X(i_1,j_2) \arrow[drr, "\mathsf{inc}^1_{i_1,j_2}"] \arrow[ddr, "X(f_1,j_2)"'] & & & \\
& & X^1(j_2) \arrow[uur, "\mathsf{inc}_{j_2}"'] & \\
& X(i_2,j_2) \arrow[ur, "\mathsf{inc}^1_{i_2,j_2}"'] & &
\end{tikzcd}$$

The left square commutes by the functoriality of X, the other squares by **FACT 1**. The three triangles commute because colimits are cocones. $X(f_1, g_1)$ is the diagonal of the left square.

Then,

$$\begin{aligned}
& z_{i_2,j_2} \circ X(f_1, g_1) \\
&= \mathsf{inc}_{j_2} \circ \mathsf{inc}^1_{i_2,j_2} \circ X(f_1, g_1) \\
&= \mathsf{inc}_{j_2} \circ \mathsf{inc}^1_{i_2,j_2} \circ X(f_1, j_2) \circ X(i_1, g_1) && \text{(Functoriality of } X\text{)} \\
&= \mathsf{inc}_{j_2} \circ \mathsf{inc}^1_{i_1,j_2} \circ X(i_1, g_1) && (X^1(j_2) \text{ is a cocone}) \\
&= \mathsf{inc}_{j_2} \circ X^1(g_1) \circ \mathsf{inc}^1_{i_1,j_1} && (\textbf{FACT 1}) \\
&= \mathsf{inc}_{j_1} \circ \mathsf{inc}^1_{i_1,j_1} && (Z \text{ is a cocone}) \\
&= z_{i_1,j_1}
\end{aligned}$$

so we can conclude that $z_{i,j} : X(i,j) \to Z$ indeed is a cocone.

8

For (c), we wish to show that $(z_{i,j})_{i\in I, j\in J}$ is the colimit cocone on X. We do this, as usual, by verifying the universal property. **Let $w_{i,j} : X(i,j) \to W$ be an arbitrary cocone on X** **8**. We want a unique map $Z \to W$ making every triangle commute (i.e. the orange shape below, for every i_1, j_1).

$$\begin{tikzcd}
X(i_1,j_1) \arrow[rd, "\mathsf{inc}^1_{i_1,j_1}"] \arrow[ddrrrr, bend left, "w_{i_1,j_1}"] \arrow[ddr, "X(f_1,j_1)"] \arrow[ddd, "X(i_1,g_1)"'] & & & & \\
& & X^1(j_1) \arrow[dr, "\mathsf{inc}_{j_1}"] \arrow[ddd, "X^1(g)"] & & \\
& X(i_2,j_1) \arrow[ur, "\mathsf{inc}^1_{i_2,j_1}"'] \arrow[ddd] & & Z \arrow[r, dashed, "\exists!"] & W \\
X(i_1,j_2) \arrow[drr, "\mathsf{inc}^1_{i_1,j_2}"] \arrow[ddr, "X(f_1,j_2)"'] & & & & \\
& & X^1(j_2) \arrow[uur, "\mathsf{inc}_{j_2}"'] & & \\
& X(i_2,j_2) \arrow[ur, "\mathsf{inc}^1_{i_2,j_2}"] \arrow[uuurrr, bend right, "w_{i_2,j_2}"'] & & &
\end{tikzcd}$$

We'll use the universal properties of colimits to define such a map.

9

To begin, observe that for all $f_1 : i_1 \to i_2$ and every j that **the following 9** commutes.

$$\begin{tikzcd}
X(i_1,j) \arrow[dr] \arrow[drr, bend left, "w_{i_1,j}"] \arrow[dd, "X(f_1,j)"'] & & \\
& X^1(j) \arrow[r, dashed, "{[w_{i,j} \mid i\in I]}"] & W \\
X(i_2,j) \arrow[ur] \arrow[urr, bend right, "w_{i_2,j}"'] & &
\end{tikzcd}$$

For the sake of notation, we'll write k_j for $[w_{i,j} \mid i \in I] : X^1(j) \to W$. **Remember its key property 10**:

$$k_j \circ \mathsf{inc}^1_{i,j} = w_{i,j} \qquad \text{for all } i, j \qquad \textbf{(FACT 3)}$$

Since we can define this for all j, we have the following diagram for any $g_1 : j_1 \to j_2$ in J,

$$\begin{tikzcd}
X^1(j_1) \arrow[dd, "X^1(g_1)"'] \arrow[dr] \arrow[drr, bend left, "k_{j_1}"] & & \\
& Z \arrow[r, dashed, "\varphi"] & W \\
X^1(j_2) \arrow[ur] \arrow[urr, bend right, "k_{j_2}"'] & &
\end{tikzcd}$$

That $k_{j_2} \circ X^1(g_1) = k_{j_1}$ can be verified using **ColimInj**: for any i,

$$\begin{aligned}
k_{j_2} \circ X^1(g_1) \circ \mathsf{inc}^1_{i,j_1} &= k_{j_2} \circ \mathsf{inc}^1_{i,j_2} \circ X(i, g_1) && (\textbf{FACT 1}) \\
&= w_{i,j_2} \circ X(i, g_1) && (\textbf{FACT 3}) \\
&= w_{i,j_1} && (w_{i,j} \text{ cocone}) \\
&= k_{j_1} \circ \mathsf{inc}^1_{i,j_1}. && (\textbf{FACT 3})
\end{aligned}$$

Therefore conclude $k_{j_2} \circ X^1(g_1) = k_{j_1}$, hence the k_js constitute a cocone on X^1. Since Z is the colimit on X^1, obtain $\varphi = [k_j \mid j \in J]$ as indicated above.

10

So it remains to show that φ has the universal property we want, i.e. it is the unique map such that $\varphi \circ z_{i,j} = w_{i,j}$ for all i, j. Observe:

$$\begin{aligned}
\varphi \circ z_{i,j} &= \varphi \circ \mathsf{inc}_j \circ \mathsf{inc}^1_{i,j} \\
&= k_j \circ \mathsf{inc}^1_{i,j} && (\varphi = [k_j \mid j \in J]) \\
&= w_{i,j}. && (\textbf{FACT 3})
\end{aligned}$$

Now, if there were $\varphi' : Z \to W$ such that $\varphi' \circ z_{i,j} = w_{i,j}$ for all i, j **11**, then that means

$$\varphi \circ \mathsf{inc}_j \circ \mathsf{inc}^1_{i,j} = \varphi' \circ \mathsf{inc}_j \circ \mathsf{inc}^1_{i,j} \qquad (\text{for all } i, j)$$

Apply **ColimInj** to get

$$\varphi \circ \mathsf{inc}_j = \varphi' \circ \mathsf{inc}_j \qquad (\text{for all } j)$$

and apply **ColimInj** once again to get

$$\varphi = \varphi'.$$

So the universal property is proven, and $z_{i,j} : X(i, j) \to Z$ is the colimit cocone on X.

11

The most straightforward way to prove (d) is to show that $\operatorname{colim}_i X^2(i)$ must also satisfy the universal property of the colimit on X, and thereby (with (c)) conclude

$$\operatorname*{colim}_j X^1(j) \cong \operatorname*{colim}_{i,j} X(i, j) \cong \operatorname*{colim}_i X^2(i)$$

since colimits are unique up to unique isomorphism.

This can be done analogously to our development of X^1. Letting Y be $\operatorname{colim}_i X^2(i)$ with inclusion maps $\mathsf{inc}_i : X^2(i) \to Y$, we can argue that the maps

$$y_{i,j} = \mathsf{inc}_i \circ \mathsf{inc}^2_{i,j} : X(i,j) \to Y$$

form a cocone by appealing to **FACT 2**, the functoriality of X, that $X^2(i)$ is a cocone, and that Y is the colimit of X^2. We can then argue that $y_{i,j}$ is the colimit of X by picking another cocone $w_{i,j} : X(i,j) \to W$ and leveraging the universal properties of Y as the colimit of X^2 and $X^2(i)$ as the colimit of $X(i,-)$ to construct a unique map $\psi : Y \to W$ such that $\psi \circ y_{i,j} = w_{i,j}$ for all i, j.

Notes:

1 A common notation convention is to write

$$\langle v_k \mid k \in K \rangle : E \to \lim_k Y(k)$$

for the unique map h such that $v_k = p_k \circ h$ for every object k in K (which we get by the universal property of limits – assuming $\lim_k Y(k)$ exists – so long as $v_k : E \to Y(k)$ is a cone on the diagram Y). $p_k : \lim_{k'} Y(k') \to Y(k)$ are the projection maps of the limit cone. This is the "indexed" generalization of the notation you've already seen for products, namely writing $\langle f, g \rangle : C \to A \times B$ for some $f : C \to A$ and $g : C \to B$. The $[w_k \mid k \in K] : \operatorname{colim}_k Y(k) \to W$ above is the dual notation for colimits.

Alternatively, you could think of these as maps:

$$\langle - \rangle : \mathsf{Cone}(Y, E) \to \mathsf{Hom}(E, \lim_k Y(k))$$
$$[-] : \mathsf{Cocone}(Y, W) \to \mathsf{Hom}(\operatorname{colim}_k Y(k), W)$$

One way to state the "bijection characterization" of the universal properties of limits and colimits (respectively) is that these functions exist and are bijections. Moreover, $\langle - \rangle$ is *natural* in E: view $\mathsf{Cone}(Y, -)$ and $\mathsf{Hom}(-, \lim_k Y(k))$ as functors $\mathbf{C}^{\mathrm{op}} \to \mathsf{Set}$ whose morphism parts are given by the appropriate notions of composition, and check that for any $s : E \to E'$ and any $(t_k)_{k \in K} \in \mathsf{Cone}(Y, E')$ in $\mathbf{C}$ that

$$\langle \mathsf{Cone}(Y, s)(t_k) \rangle = \mathsf{Hom}(s, \lim_k Y(k)) \ \langle t_k \mid k \in K \rangle$$

i.e. that $\langle t_k \rangle \circ s$ is the unique map h such that $t_k \circ s = p_k \circ h$ for all k. Likewise, $\mathsf{Cocone}(Y, -)$ and $\mathsf{Hom}(\operatorname{colim}_k Y(k), -)$ are functors $\mathbf{C} \to \mathsf{Set}$ with $[-]$ a natural isomorphism between them.

2 This is a good way to write proofs: identify some styles of reasoning you're going to use over and over, isolate and name them, and then refer to them (instead of having to walk through the same logic over and over).

3 Here I write a hyphen to indicate where the argument goes, which is useful for "partially applying" multi-argument functors. For instance, we can take the functor

$$\mathsf{Hom} : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathsf{Set}$$

and supply just one argument X to get the functors

$$\mathsf{Hom}(X, -) : \mathbf{C} \to \mathsf{Set}$$

and

$$\mathsf{Hom}(-, X) : \mathbf{C}^{\mathrm{op}} \to \mathsf{Set}.$$

Of course, we write $\mathsf{Hom}(X, Y)$ instead of $\mathsf{Hom}(-, Y)(X)$. There's some things to check to make sure that this is legitimate, but it is.

Note that for $f : i \to i'$ in I, $X(f, j)$ is the same thing as $X(f, 1_j)$, i.e. we take $(f, 1_j) : (i, j) \to (i', j)$ as a morphism in $I \times J$ and apply the functor X to it. This is also a common notational convention (sometimes it gets tedious to write all the 1's).

4 Whenever you're asked to supply a functor, **always do this**. For any functor you define, you need to make clear what the object and morphism parts are and prove functoriality (that it preserves identities and composition). You don't have to go into detail if it's trivial, but you need to mention it . . . otherwise how do I know that you understand that these are the necessary components/requirements of a functor?

5 A very common mistake I see is forgetting to do this. **You cannot apply the universal properties of (co)limits to things that are not (co)cones.** Just because you have some data of the right type (in this case, an I-indexed collection of maps $X(i, j_1) \to X^1(j_2)$), that does not make it a cocone: it has to satisfy the commutativity condition.

6 Pro tip: name your facts! Makes it easier to refer to later.

7 I was relatively lenient on grading this, since it is very similar. But in the future, you're expected to do what I do here: indicate what the key facts & steps are in the proof.

8 This is how basically every proof of a universal property should start.

9 It's important to keep your diagrams manageable. I saw plenty of people try to draw the entire setup in one diagram. . . for a complex enough problem (like this one), that's going to confuse more than help. Depict the aspect you're discussing at the moment, and leave the other details for other diagrams.

10 I'm forgetful and easily distracted. Tell me directly what the key facts are.

11 This reminds me of another common mistake. Remember: universal properties usually have the conclusion "there exists a unique ____ such that ______". Don't forget the "such that ______"! There's not a unique map $\varphi : Z \to W$, full stop. There's a unique $\varphi : Z \to W$ ***such that*** $\varphi \circ z_{i,j} = w_{i,j}$.

Problem:

Let I and J be two small categories and $\mathbf{C}$ a category with colimits of shape I and limits of shape J. We fix a diagram
$$X : I \times J \to \mathbf{C}$$

(a) Using the universal property of limits and colimits, construct a map
$$\operatorname*{colim}_{i\in I} \lim_{j\in J} X(i,j) \to \lim_{j\in J} \operatorname*{colim}_{i\in I} X(i,j)$$

(b) Find an example of a diagram $X : I \times J \to \mathbf{C}$ for which this map is not an isomorphism.

Solution:

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Let's define $X^1 : J \to \mathbf{C}$ as in the previous problem: each $j \in J$ gets sent to $\operatorname{colim}_i X(i,j)$ and each $g_1 : j_1 \to j_2$ gets sent to the unique map
$$\left[\mathsf{inc}^1_{i,j_2} \circ X(i,g_1) \;\middle|\; i \in I\right] : \operatorname*{colim}_i X(i,j_1) \to \operatorname*{colim}_i X(i,j_1)$$
which, as established above, satisfies the key property that, for any $i \in I$,
$$X^1(g_1) \circ \mathsf{inc}^1_{i,j_1} = \mathsf{inc}^1_{i,j_2} \circ X(i,g_1). \qquad \textbf{(FACT 1)}$$
We'll do the dual thing for the other argument, putting
$$\begin{aligned} X^2 &: I \to \mathbf{C} \\ &: i \mapsto \lim_j X(i,j) \end{aligned}$$
and then noticing that for any j and any $f_1 : i_1 \to i_2$ in I that the maps
$$X(f_1,j) \circ p_{i_1,j} \;:\; \lim_j X(i_1,j) \to X(i_2,j)$$
form a cone on $X(i_2,-)$, hence we can put
$$X^2(f_1) = \langle X(f_1,j) \circ p_{i_1,j} \;\mid\; j \in J\rangle : \lim_j X(i_1,j) \to \lim_j X(i_2,j)$$
which have a dual property to **FACT 1**:
$$p_{i_2,j} \circ X^2(f_1) = X(f_1,j) \circ p_{i_1,j}. \qquad \textbf{(FACT 4)}$$

$$\begin{tikzcd}
& X(i_1,j) \arrow[r, "X(f_1,j)"] & X(i_2,j) & \\
\lim_j X(i_1,j) \arrow[ur, "p_{i_1,j}"] \arrow[rrr, "X^2(f_1)"'] & & & \lim_j X(i_2,j) \arrow[ul, "p_{i_2,j}"']
\end{tikzcd}$$

Let us also recall that for all i, j, all $f_1 : i_1 \to i_2$ and all $g_1 : j_1 \to j_2$ that

$$\begin{aligned} \mathsf{inc}^1_{i_2,j} \circ X(f_1, j) &= \mathsf{inc}_{i_1,j} && \text{(Colimits are cocones)} \\ X(i, g_1) \circ p_{i,j_1} &= p_{i,j_2} && \text{(Limits are cones)} \end{aligned}$$

Now, our goal is to construct a map

$$\operatorname*{colim}_i X^2(i) \to \lim_j X^1(j).$$

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Start by considering these maps, for any i_1, j_1:

$$\lim_j X(i_1, j) \xrightarrow{p_{i_1,j_1}} X(i_1, j_1) \xrightarrow{\mathsf{inc}^1_{i_1,j_1}} \operatorname{colim}_j X(i, j_1)$$

Call their composite $h_{i_1,j_1} = \mathsf{inc}^1_{i_1,j_1} \circ p_{i_1,j_1}$. Notice that these form a cone on X^1: for any $g_1 : j_1 \to j_2$ and any i, the triangle

$$\begin{tikzcd} & X^1(j_1) \arrow[dd, "X^1(g_1)"] \\ X^2(i) \arrow[ur, "h_{i,j_1}"] \arrow[dr, "h_{i,j_2}"'] & \\ & X^1(j_2) \end{tikzcd}$$

commutes:

$$\begin{aligned} & X^1(g_1) \circ h_{i,j_1} \\ &= X^1(g_1) \circ \mathsf{inc}^1_{i,j_1} \circ p_{i,j_1} \\ &= \mathsf{inc}^1_{i,j_2} \circ X(i, g_1) \circ p_{i,j_1} && (\textbf{FACT 1}) \\ &= \mathsf{inc}^1_{i,j_2} \circ p_{i,j_2} && \text{(Limits are cones)} \\ &= h_{i_2,j_2}. \end{aligned}$$

Therefore, we obtain for each i a map

$$k_i = \langle h_{i,j} \mid j \in J \rangle : X^2(i) \to \lim_j X^1(j).$$

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These maps form a cocone on X^2: for each $f_1 : i_1 \to i_2$, the triangle

$$\begin{tikzcd}
X^2(i_1) \arrow[dd, "X^2(f_1)"'] \arrow[dr, "k_{i_1}"] & \\
& \lim_j X^1(j) \\
X^2(i_2) \arrow[ur, "k_{i_2}"'] &
\end{tikzcd}$$

commutes:

$$\begin{aligned}
& k_{i_2} \circ X^2(f_1) \\
& = \langle h_{i_2,j} \mid j \in J \rangle \circ X^2(f_1) \\
& = \langle h_{i_2,j} \circ X^2(f_1) \mid j \in J \rangle && (\textbf{12}) \\
& = \langle \mathsf{inc}^1_{i_2,j} \circ p_{i_2,j} \circ X^2(f_1) \mid j \in J \rangle \\
& = \langle \mathsf{inc}^1_{i_2,j} \circ X(f_1, j) \circ p_{i_1,j} \mid j \in J \rangle && (\textbf{FACT 4}) \\
& = \langle \mathsf{inc}^1_{i_1,j} \circ p_{i_1,j} \mid j \in J \rangle && (\text{Colimits are cocones}) \\
& = \langle h_{i_1,j} \mid j \in J \rangle \\
& = k_{i_1}.
\end{aligned}$$

Therefore we have obtained a cocone on X^2 whose apex is $\lim_j X^1(j)$, so there is a unique map

$$[k_i \mid i \in I] : \operatorname*{colim}_i X^2(i) \to \lim_j X^1(j).$$

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For (b), the **easiest counterexample is to take** $I = J = \emptyset$ **13** and $\mathbf{C} = \mathsf{Set}$ (or $\mathsf{Set}_{<\omega}$). Recall that the colimit over an empty diagram is an initial object, which is $\emptyset$ in Set, i.e.

$$\emptyset = \operatorname*{colim}_i X^2(i).$$

Dually, the limit of an empty diagram is a terminal object, which is $\{\star\}$ in Set, so

$$\{\star\} = \lim_j X^1(j).$$

The map we obtained in the previous part is the unique map $\emptyset \to \{\star\}$, but there cannot be a map $\{\star\} \to \emptyset$, hence this cannot be an isomorphism.

Notes:

12 The general fact is this: for any diagram $Y : K \to \mathbf{C}$ such that $\lim_k Y(k)$ exists, and for any cone $z_k : Z \to Y(k)$ and any map $f : Z' \to Z$,

$$\langle z_k \mid k \in K \rangle \circ f = \langle z_k \circ f \mid k \in K \rangle$$

You proved the special case of binary products (i.e. $K = \{0, 1\}$) on Homework 4. This is again a consequence of the injectivity of the map $\mathsf{Cone}(Y, Z') \to \mathsf{Hom}(Z', \lim_k Y(k))$, and is the key fact behind the proof that $\langle - \rangle$ is natural in its cone argument.

13 This is a generally-applicable lesson for math: if you need a counterexample, always try the empty set!