Category Theory (80-413/713) F20 HW5, Exercise 2 Solution

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Problem:

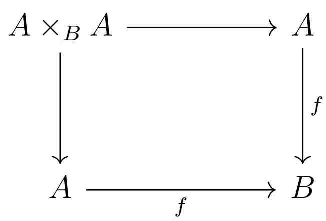
(a) Given a map of sets $f : A \to B$, prove that $A \times_B A \subseteq A \times A$ is an equivalence relation on A. This relation is called the *kernel* of f.

(b) Let $R \subseteq G_0 \times G_0$ be the kernel of the map $q : G_0 \to \pi_0(G)$ and $k : G_0 \to Q = G_0/R$ the quotient map. Use the universal property of $\pi_0(G)$ as the coequalizer of $s, t : G_1 \rightrightarrows G_0$ to construct a map $\pi_0(G) \to Q$ and use this to prove that $k : G_0 \to Q$ is also a coequalizer of $s, t : G_1 \rightrightarrows G_0$.

Solution:

1

For (a), we're given this data:



Recall that pullbacks **in Set** [1] are given as the set of pairs which have the same image under the functions of the span:

$$A \times_B A = \{(a, a') \in A \times A \ : \ f(a) = f(a')\}.$$

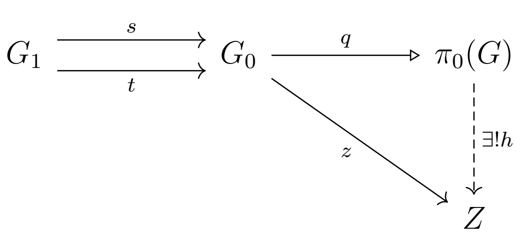
Let's write $a \sim a'$ to indicate that $(a, a') \in A \times_B A$. We need to show that $\sim$ is an equivalence relation, that is, reflexive, symmetric, and transitive.

2

Each of these are straightforward [2]. Of course, for any $a \in A$, we have $f(a) = f(a)$, so $a \sim a$, giving reflexivity. If $a \sim a'$, then $f(a) = f(a')$, hence $f(a') = f(a)$, hence $a' \sim a$, yielding symmetry. Finally, for reflexivity, notice that $f(a) = f(a')$ and $f(a') = f(a'')$ obviously implies $f(a) = f(a'')$. So we're done.

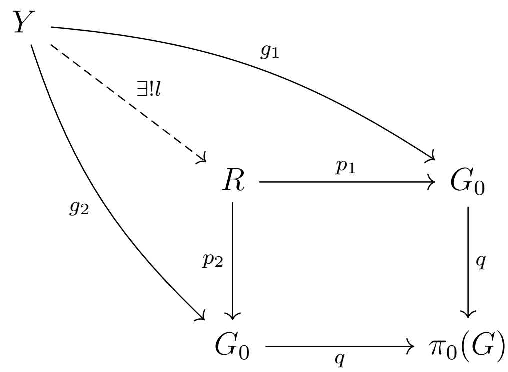
3

Moving on to (b), **let's review the relevant parts of our setup** [3]. First, we have that $q : G_0 \to \pi_0(G)$ is the coequalizer of s and t, i.e. **for any other $z : G_0 \to Z$ with $zs = zt$, there's a unique h making this diagram commute** [4].



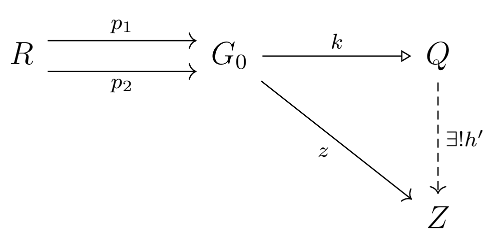
(Diagram 1)

We also have that R is the kernel of $q : G_0 \to \pi_0(G)$, so if there's **any set Y and any maps $g_1, g_2 : Y \rightrightarrows G_0$ such that $qg_1 = qg_2$** [5], we have a map l making this diagram commute.



(Diagram 2)

4 Finally, we also have Q defined as G_0/R, the quotient of G_0 by the relation R from the previous paragraph (the one defined by Diagram 2). Remember that this operation – taking the quotient of a set by an equivalence relation – **is the same thing as taking a coequalizer in Set** [6]. Specifically, the following coequalizer diagram commutes for any z.



(Diagram 3)

We wish to show that $k : G_0 \to Q$ is the coequalizer of s and t.

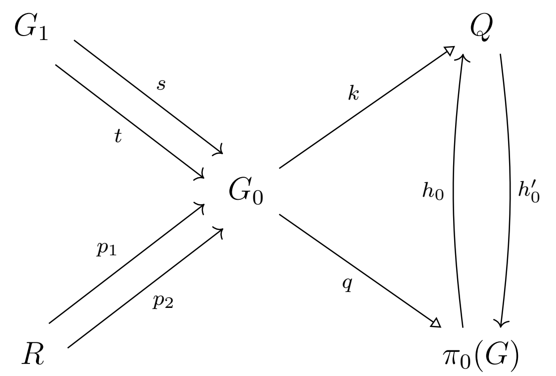
5 We first must show that k ***coequalizes* s and t, i.e. that $ks = kt$** [7]. Notice that $qs = qt$ by Diagram 1. So $s, t : G_1 \rightrightarrows G_0$ is a cone on the cospan in Diagram 2, hence **we obtain a map $l : G_1 \to R$ such that $p_1 l = s$ and $p_2 l = t$** [8]. In other words, for every edge $e \in G_1$, we have that $l(e) = (s(e), t(e)) \in R$. Since R-related things get identified by k, we have that

$$k(s(e)) = k(t(e)) \quad \text{for all } e \in G_1$$

hence $ks = kt$.

6 Now, since k coequalizes s and t, we get a unique map $\pi_0(G) \to Q$ which we'll call h_0 [9] such that $k = h_0 q$ (Diagram 1). But notice by the commutativity of the square in Diagram 2 that q coequalizes $p_1, p_2 : R \rightrightarrows G_0$, and thus by Diagram 3 we get a unique

$h_0' : Q \to \pi_0(G)$ such that $q = h_0' k$.



(Diagram 4)

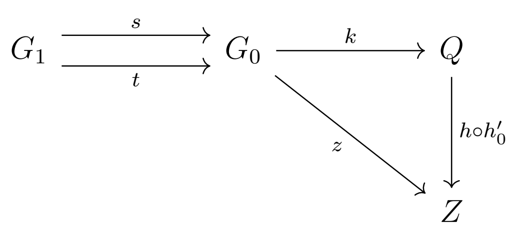
Let's show that h_0 and h_0' are inverses (hence isomorphisms) [10]. Observe that

$$h_0 h_0' k = h_0 q = k \qquad \text{and} \qquad h_0' h_0 q = h_0' k = q.$$

But the universal property of k as the coequalizer of p_1, p_2 states that there is a *unique* $h' : Q \to Q$ such that $k = h'k$, and 1_Q satisfies this equation. So the left equality above implies $h_0 h_0' = 1_Q$. Identical logic about q as the coequalizer of s, t, plus the equality on the right, will give that $h_0' h_0 = 1_{\pi_0(G)}$.

7

You could've concluded here with "since Q is isomorphic to $\pi_0(G)$, the coequalizer of s, t, Q is a coequalizer of s, t", since isomorphic objects are regarded as the same. But I'll prove the universal property directly using this isomorphism, just for further reference. Pick an arbitrary function $z : G_0 \to Z$ such that $zs = zt$. By Diagram 1, obtain a unique $h : \pi_0(G) \to Z$ such that $z = hq$. We now claim that **hh_0' is the unique map making the following diagram commute.** [11]



(Diagram 5)

The commutativity of the triangle is simple: since $q = h_0' k$, we get that $hh_0' k = hq = z$. For uniqueness, if there were another map $r : Q \to Z$ with $z = rk$, then observe $rh_0 : \pi_0(G) \to Z$ satisfies

$$z = rh_0 q$$

because $k = h_0 q$. But remember that h was the unique map such that $z = hq$, hence $h = rh_0$. But then

$$r = r1_Q = rh_0 h_0' = hh_0'$$

so $r = hh_0'$ as desired.

Notes:

1 This is not true (or perhaps doesn't even make sense) in other categories! If you're asked about a pullback (or any abstract category-theoretic construction, e.g. products, coequalizers, etc.) in an arbitrary category, all you have to work with is the universal property. You cannot use the formulas that work in Set (in this case, the pullback formula $A \times_B C = \{(a, c) \in A \times C \ : \ f(a) = g(c)\}$). In this problem, we know we're working in Set, so it's allowed. The Set (co)limit formulas are also allowed in $\mathsf{Set}_{<\omega}$, $\mathsf{Set}_{\leq\omega}$, etc., if they indeed exist in the category.

2 The proofs of these are all a bit silly. But if the question asks you to prove something is an equivalence relation, please do at least mention reflexivity, symmetry, and transitivity, even if only to dismiss each as trivial.

3 For a complex problem like this, I highly recommend starting by "reminding" yourself (and me, your grader) what data you're working with, what the salient properties are (e.g. which universal properties will you appeal to in your proof), and what you're trying to show. This will both (a) help you solve the problem more effectively (because you'll have all the relevant facts laid out in front of you), and (b) make it more clear to me that you understand what's happening (which is one of the main things I'm looking for). This isn't to say that you should just copy the problem statement into your solution: the main goal here is to make *explicit* the facts which were *implicit* in the problem statement.

4 Notice the form of this statement: for any z coequalizing s and t, there's a unique h making the diagram commute. Throughout the proof, I will justify claims "by Diagram 1" accordingly: I'll specify a z such that $zs = zt$, and use Diagram 1 to obtain the corresponding h, which I can be assured is the *unique* h making the diagram commute (which is sometimes important).

Also notice that the arrow tip on the q in the diagram is a little bit different than the other arrows. This triangular arrow tip indicates that the morphism in question is the coequalizer of the parallel pair of arrows, i.e. it is a regular epimorphism.

5 A "cone on the cospan in Diagram 2"

6 So what I do here is interpret the quotienting operation as a universal property, specifically the universal mapping property of a coequalizer. This is more convenient than the set-theoretic formulation of quotients (see **10**), because universal properties are convenient for constructing maps, proving the commutativity of diagrams, and proving other universal properties.

7 Recall this terminology: "k *coequalizes* s and t" means just that $ks = kt$. "k is *the coequalizer* of s and t" means that k coequalizes s and t, and, moreover, does so universally: it is the initial object in the category of cocones (in Set) over the diagram $s, t : G_1 \rightrightarrows G_0$.

8 Note the little convention I've established: Diagram 1 produces hs, Diagram 3 produces h's, and Diagram 2 – like the Pittsburgh Pirates – produces ls. Those little notation rules will help save you from getting lost in the proof.

Note that I'm using the p_1 and p_2 defined as usual from Set-pullbacks, which decomposes pairs $(x, y) \in R \subseteq G_0 \times G_0$ into their components, x and y.

9 Note that I'm using h_0 to refer to *this specific* map $\pi_0(G) \to Q$ produced by Diagram 1. I'll still leaving the letter h to generically denote the maps produced by Diagram 1 for whatever $z : G_0 \to Z$ coequalizes s and t.

10 I think the category-theoretic argument above is a lot slicker (and the kind of argument you should produce for this course). But if you like a more explicit set-theoretic proof, here I suppose h_0 is given by the universal property of q (a category-theoretic argument) but I show the existence of h_0' and that it's the inverse of h_0 using a set-theoretic argument. This was the proof I presented in office hours on 14 October.

8

Recall that the elements of Q are R-equivalence classes of the form $[x]_R$ for some $x \in G_0$. We claim that we can define $h_0' : Q \to \pi_0(G)$ by

$$h_0'([x]) = q(x)$$

and that this h_0' is an inverse of h_0. This will prove that Q and $\pi_0(G)$ are isomorphic, hence Q is the coequalizer of s, t.

9

First, we must see that h_0' is well-defined, i.e. our definition of h_0' is not dependent on which equivalence class representative we pick. So it suffices to show for any $x, y \in G_0$ that if $k(x) = k(y)$ then $q(x) = q(y)$. But if $k(x) = k(y)$, this means that $(x, y) \in R$, and, by Diagram 2,

$$q(x) = q(p_1(x, y)) = q(p_2(x, y)) = q(y).$$

So h_0' is well-defined. To see that h_0' is indeed the inverse of h_0, observe:

$$\begin{aligned} h_0 h_0'([x]) &= h_0(q(x)) && \text{(defn. of } h_0'\text{)} \\ &= k(x) && (k = h_0 q) \\ &= [x] && \text{(defn. of } k\text{)} \end{aligned}$$

$$\begin{aligned} h_0' h_0(q(x)) &= h_0'(k(x)) \\ &= h_0'([x]) \\ &= q(x) \end{aligned}$$

where we assume that all elements of $\pi_0(G)$ are of the form $q(x)$ for some $x \in G_0$ since q is surjective.

11 To prove the universal property, we're supplying hh_0' as the map $Q \to Z$ fulfilling the necessary conditions. We have $h : \pi_0(G) \to Z$ from the universal property of q as a coequalizer and $h_0' : Q \to \pi_0(G)$ from the universal property of k as a coequalizer, so hh_0' exists and has the proper domain and codomain.