Category Theory (80-413/713) F20 HW4, Exercise 2 Solution

Problem:

> Recall that an order relation $\leq$ on a set P is total if, for any x and y in P, it is true that $x \leq y$ or $y \leq x$. By viewing a total order as a category, show that the maximal element of a finite family $(x_1, \ldots, x_n)$ satisfies the definition of the colimit of the family.

Solution:

1

Consider a total order P viewed as a category. A finite family $x_1, \ldots, x_n$ of elements of **P is the same thing as a functor** [1]

$$x : I \to P$$

where $I = \{1, \ldots, n\}$ is the category with objects $1, \ldots, n$ and only identity morphisms. Since $\{x_1, \ldots, x_n\}$ is a finite subset of a total order, **there must exist some $x_M \in \{x_1, \ldots, x_n\}$ such that $x_i \leq x_M$ for all $i \in I$** [2]. We wish to show that x_M is the colimit object in P of the diagram x.

2

Let's first see that x_M constitutes a cocone on this diagram. **Since $x_i \leq x_M$ for all i and P is a *thin* category, we have that $\mathsf{Hom}_P(x_i, x_M) = \{\star\}$ for all i** [3], i.e. for each $i \in I$ we have a map $x_i \to x_M$, which we'll denote $\star_i$. Since there are no non-identity morphisms in I, there are no commutative triangles to check to show that x_M is indeed a cocone on x.

3

Finally, we show that this is the initial cocone on x. Suppose **$(z_i : x_i \to z)_{i \in I}$ is a cocone on x** [4]. Then, for all i we have that $\mathsf{Hom}_P(x_i, z)$ is inhabited, and thus **$x_i \leq z$ for all i. In particular, $x_M \leq z$, and so we have a morphism $h : x_M \to z$** [5]. Since P is a poset (and all its nonempty hom-sets are singletons), we get that **every triangle commutes** [6]. In particular, **$h \circ \star_i = z_i$ for all i** [7]. Finally, we see that **h is the unique map with this property** [8], again because $\mathsf{Hom}_P(x_M, z)$ is a singleton. Thus we have proved that x_M satisfies the universal property of colimits, and is therefore ***the* colimit of x** [9].

Notes:

[1] Here I was looking for you to either (a) interpret the finite family as a functor from a finite discrete category (a finite set) – which is what I do here –, or equivalently (b) interpret it as a "diagram in P" (a given collection of objects and morphisms within P, in this case just finitely-many objects). The family must be understood as one of these in order for it to make sense when we say "the colimit of this family", since *functors* and *diagrams* are the kinds of things *of which we can take the colimit.*

2 This is a basic fact of order theory, which you were not obliged to prove. It is, however, interesting to think about how you'd prove this. One way involves an induction on n (the size of the finite set), where x_M is defined recursively using the binary max operation, which has the property that for all $y, y' \in P$, $y \leq \max(y, y')$ and $y' \leq \max(y, y')$. This is where it's important that P is a total order, since $\max(y, y')$ would not always be defined if P were allowed to be an arbitrary partial order.

3 I assigned a point to correctly recognizing that it's necessary to prove x_M is a cocone in order to prove x_M a colimit. While some students successfully argued x_M is a colimiting cocone all at once, the most successful solutions explicitly proved cocone first, then colimit, hence I did that here.

This sentence ("Since $x_i \leq x_M \ldots$") establishes the data of a cocone: a map from each object in the diagram (the objects $x_1, \ldots, x_n$) to the apex of the cocone, x_M in this case. All that's left to check is that, for each morphism $u : i \to j$ in I, the triangle

$$\begin{tikzcd} x_i \arrow[dd, "x(u)"'] \arrow[dr, "\star_i"] & \\ & x_M \\ x_j \arrow[ur, "\star_j"'] & \end{tikzcd}$$

commutes. As the rest of Paragraph 2 explains, this is trivial.

4 Remember: the "universal" in "universal property of colimits" refers to the fact that *all* cocones on the diagram "factor through" the colimit. In most cases, proving a statement about *all* cocones is done by picking an arbitrary cocone and showing that it satisfies the statement. Making it explicit that you're taking some arbitrary cocone (like I've done here) demonstrates that you understand this point (and more broadly that you get the definition of colimit). When proving or using any universal property, making it clear what you're "universalizing" over is a good idea.

5 This is the first of three key steps in Paragraph 3. We have our arbitrary cocone with apex z, and, if x_M is inital among cocones we must have a unique map $h : x_M \to z$ which commutes with all the cocone maps of x_M and z. Step One: prove that there is a map h; Step Two: show that it commutes with the cocone maps; and Step Three: argue that it's the unique map to do so.

Step One is to get the map, and we're able to get that from x_M's location *in* the diagram (it is one of the x_i) and the fact that z is a cocone.

6 This fact will save us a lot of tedious triangle-checking: in a category where all the hom-sets are either empty or singletons, *every* triangle you can draw (i.e. every triple of maps $a : x \to y$, $b : y \to z$, $c : x \to z$ for some x, y, z) commutes ($c = b \circ a$). This is due to the fact that c and $b \circ a$ have the same domain and codomain

$$c \in \mathsf{Hom}(x, z) \qquad b \circ a \in \mathsf{Hom}(x, z)$$

and we assumed that $\mathsf{Hom}(x, z)$ is either $\emptyset$ or $\{\star\}$, hence $c = \star = b \circ a$. You didn't need to argue this (just note it).

I use the term "thin category" for this kind of category (one whose hom-sets are at most singletons), following https://ncatlab.org/nlab/show/thin+category.

7 This is the second key step of Paragraph 3: prove that the h we got commutes with the cocone maps of x_M and z, in the sense that $z_i = h \circ \star_i$ for all i.

8 This is the third key step of Paragraph 3: prove that h is the unique map making all these triangles commute. We again use the fact we discussed above (**6**).

9 Recall that we generally ignore the difference between objects which are isomorphic (and especially ones which are isomorphic via a unique isomorphism), and thereby speak of *the* colimit of a given diagram.