An Introduction to Directed Equality

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Harvard PL Seminar

This talk

- Slides: jacobneu.com/research/slides/harvard-202510.pdf

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- Preprint about the *dual-context* aspect of the type theory forthcoming

Identity types

$$\frac{s : A \qquad t : A}{\mathrm{Id}(s, t)\ \textbf{type}}$$

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- Encodes the proposition that s equals t as a type:
  - a term $p : \mathrm{Id}(s, t)$ is a *witness* or *proof* that s equals t
  - if s *does not* equal t, this is represented by $\mathrm{Id}(s, t)$ being empty

Properties of equality

- Reflexivity asserted axiomatically:

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$$\frac{p : \mathsf{Id}(s, t)}{p^{-1} : \mathsf{Id}(t, s)}$$

- Transitivity can be proved:

$$\frac{p : \mathsf{Id}(s, t) \quad q : \mathsf{Id}(t, u)}{p \cdot q : \mathsf{Id}(s, u)}$$

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- Congruence can be proved:

$$\frac{x : A \vdash B(x)\ \textbf{type} \quad p : \mathsf{Id}(s, t) \quad b : B(s)}{\mathsf{tr}_B\ p\ b : B(t)}$$

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Wild idea: What if we dropped symmetry?

$$\frac{s : A \qquad t : A}{\mathrm{Hom}(s, t)\ \textbf{type}}$$

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$$\frac{p : \mathrm{Hom}(s, t) \quad q : \mathrm{Hom}(t, u)}{p \cdot q : \mathrm{Hom}(s, u)}$$

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0 What is directed equality?

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- Encodes the proposition that s *becomes* t

Interpretation 0:
Rewriting

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  - Free/Left: Id is the equivalence relation generated by Hom (this is the same as $\mathrm{Id}(s, t) = \exists x.\mathrm{Hom}(s, x) \times \mathrm{Hom}(t, x)$ if Hom is nice)

Interpretation 1: Directed algebraic topology

- Think of a type A as a *space* (in the sense of algebraic topology); the terms $s : A$ are the points of the space

s $\qquad\qquad\qquad\qquad$ t

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$$\begin{tikzcd} s \arrow[r, bend left, "p"] & t \arrow[r, bend left, "q"] & u \end{tikzcd}$$

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$$\begin{tikzcd} s \arrow[r, bend left, "p"] & t \arrow[l, bend left, "p^{-1}"] \end{tikzcd}$$

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$$\begin{tikzcd} s \arrow[loop left] & & & t \arrow[loop right] \end{tikzcd}$$

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  - Asymmetry: cannot (in general) reverse paths (directedness!)

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$$\begin{tikzcd} s \arrow[rr, bend left, "p"] & & t \arrow[ll, bend left, dashed, red] \end{tikzcd}$$

Interpretation 2: Computer processes

- Think of a type A as the state space of some computer system, with terms $s, t : A$ as the states

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$$
\begin{tikzcd}
s \arrow[r, bend left, "p"] & t
\end{tikzcd}
$$

- Think of a type A as the state space of some computer system, with terms $s, t \colon A$ as the states
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- Interpret the properties of directed equality

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$$
\begin{tikzcd}
s \arrow[r, bend left, "p"] \arrow[rr, bend right, "p;q"'] & t \arrow[r, bend left, "q"] & u
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$$
\begin{tikzcd}
s \arrow[r, bend left, "p"] & t \arrow[l, bend left, dashed]
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Holes are concurrency deadlocks

Suppose we have two resources X and Y that are being accessed by concurrent threads. To avoid conflicts, a thread will lock a resource while using it, and prevent other threads from locking it (until unlocked).

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Suppose we have two threads, running:

- **A**: `lock X; lock Y; unlock Y; unlock X`
- **B**: `lock Y; lock X; unlock X; unlock Y`

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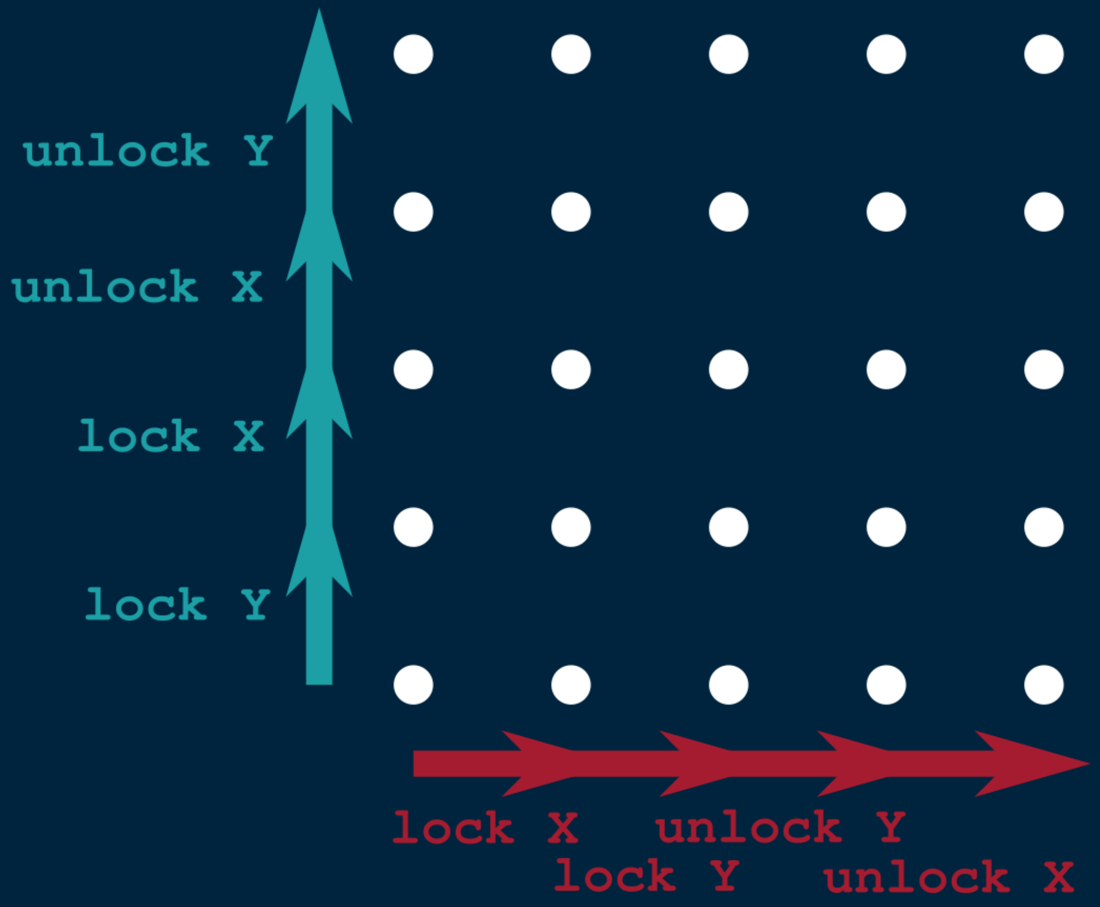
- **B**: `lock Y; lock X; unlock X; unlock Y`

Depending on how these are interleaved, we could have a deadlock


unlock Y
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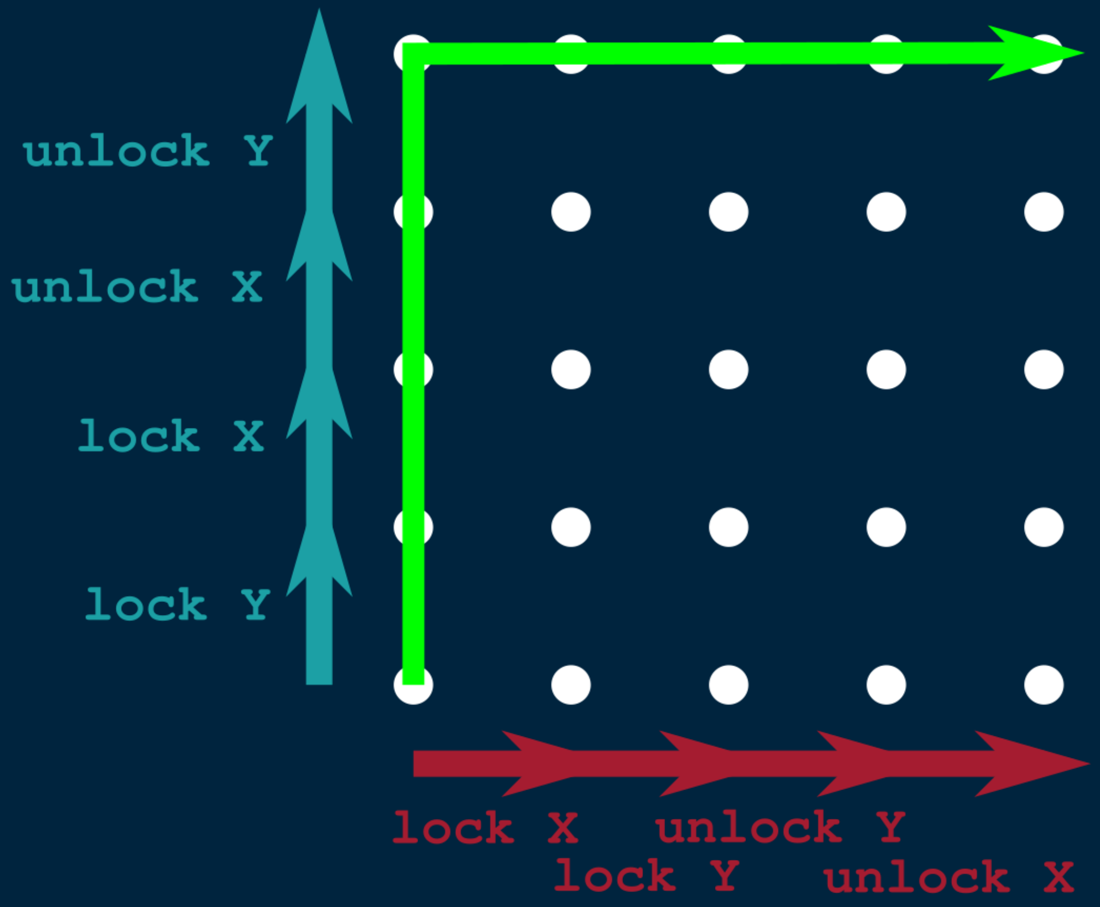
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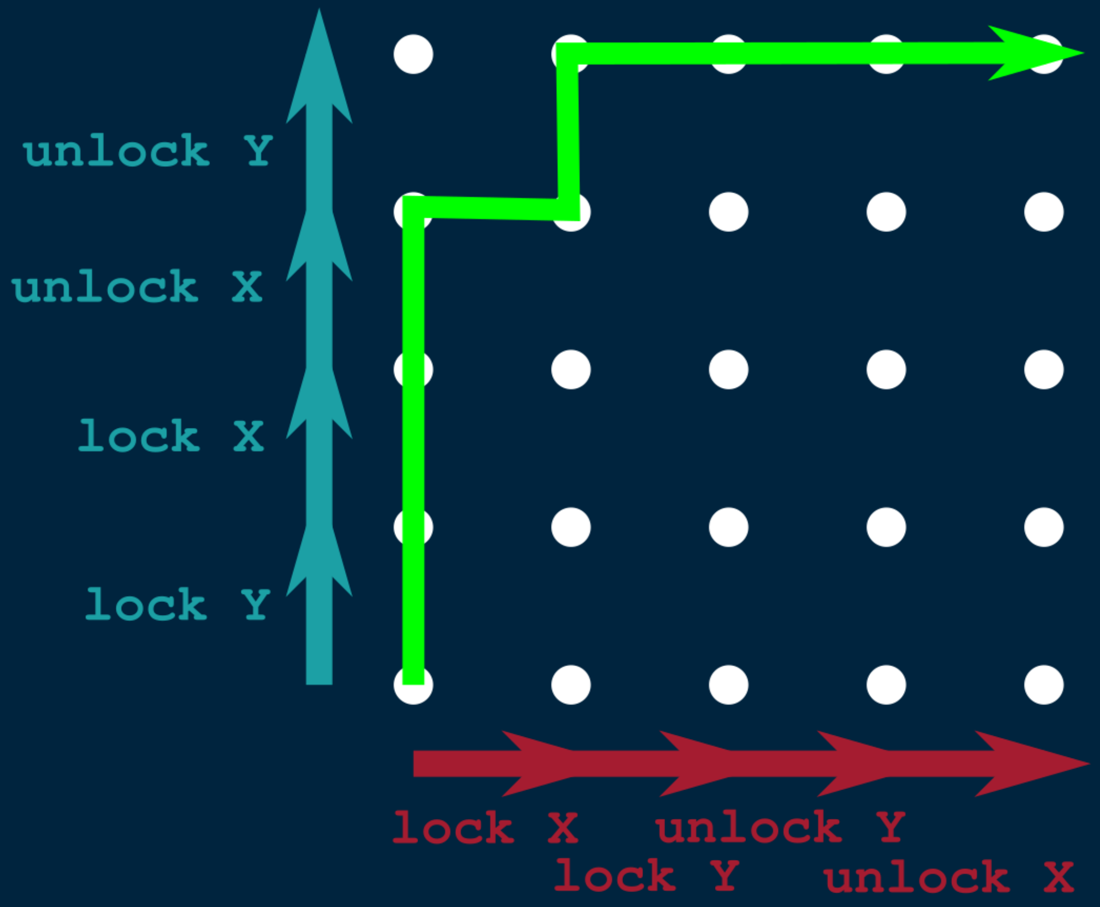
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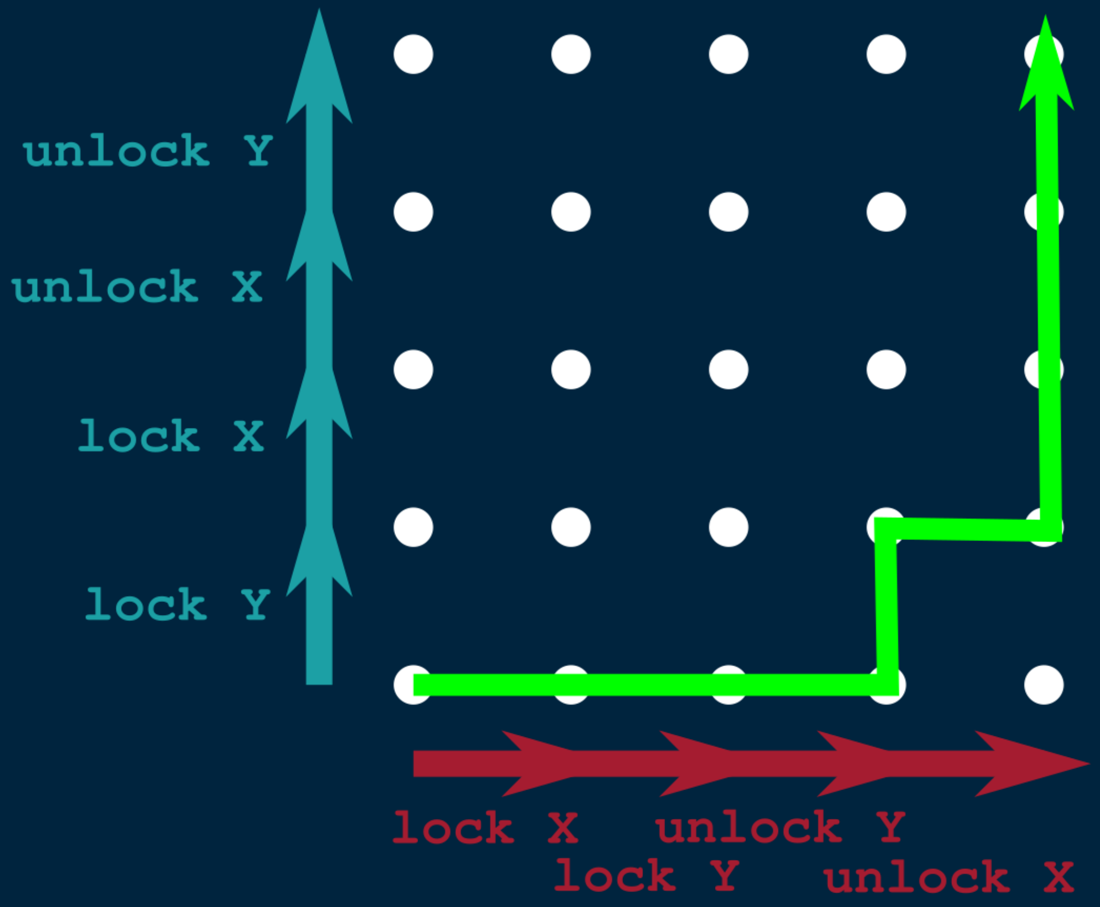
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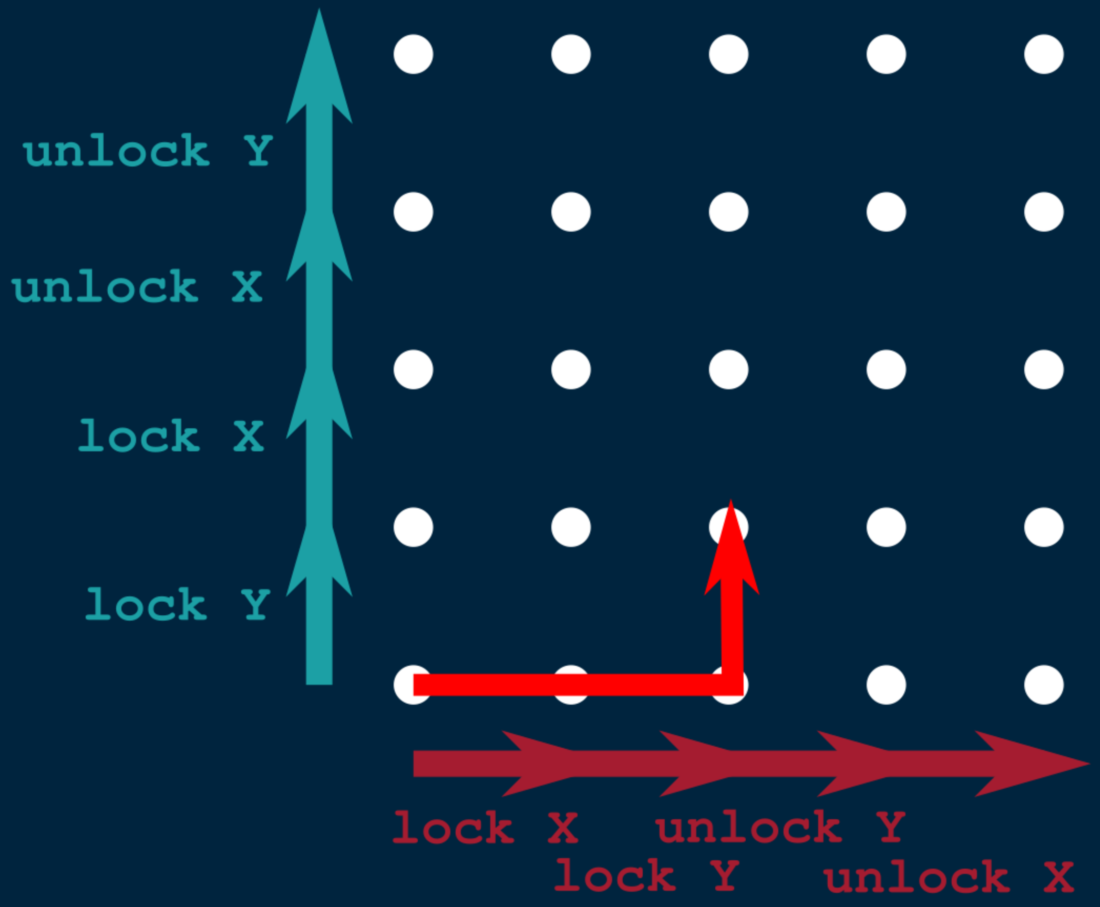
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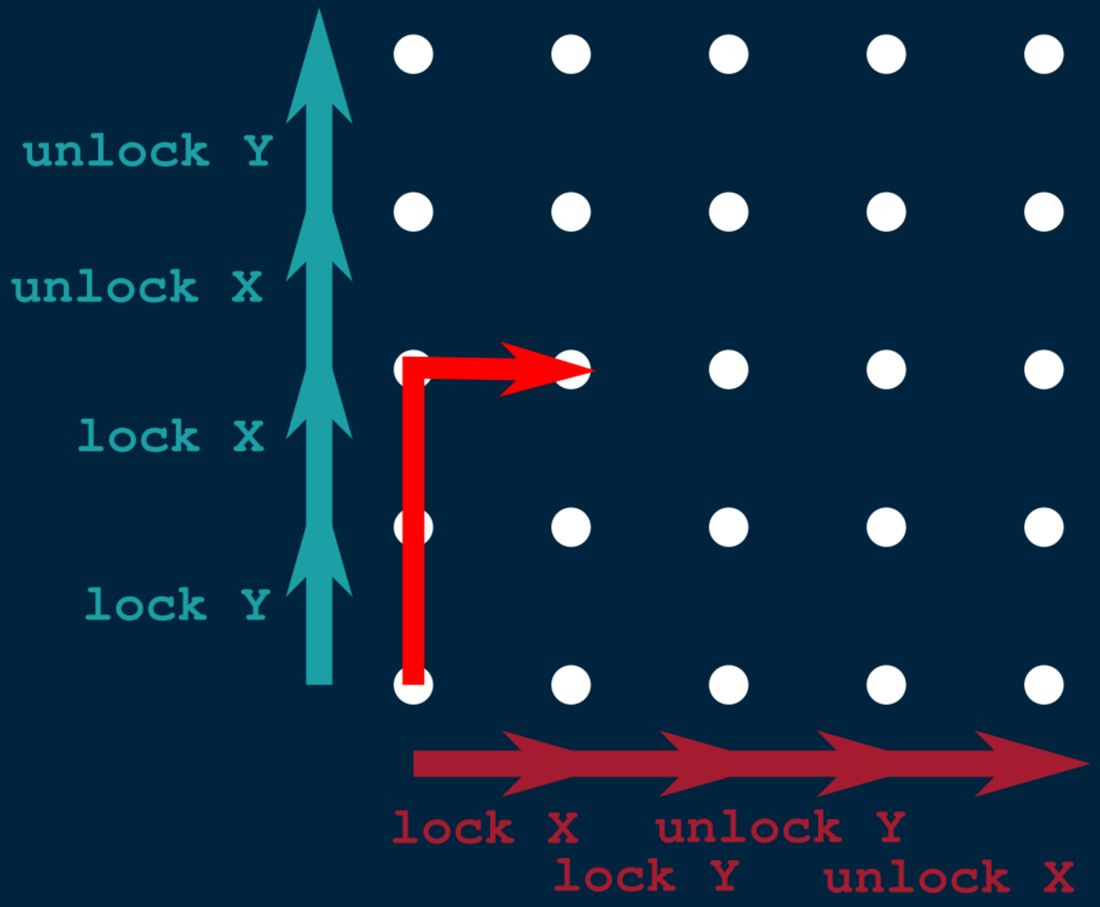
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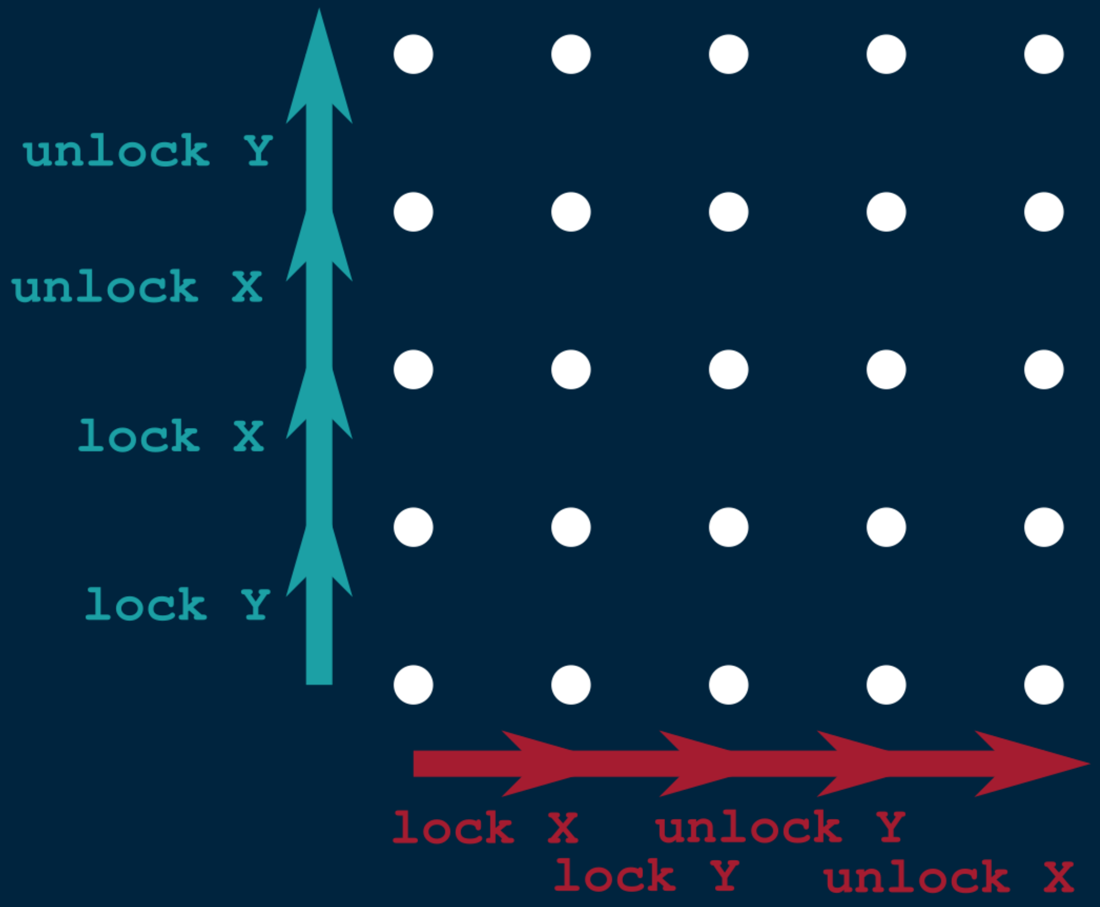
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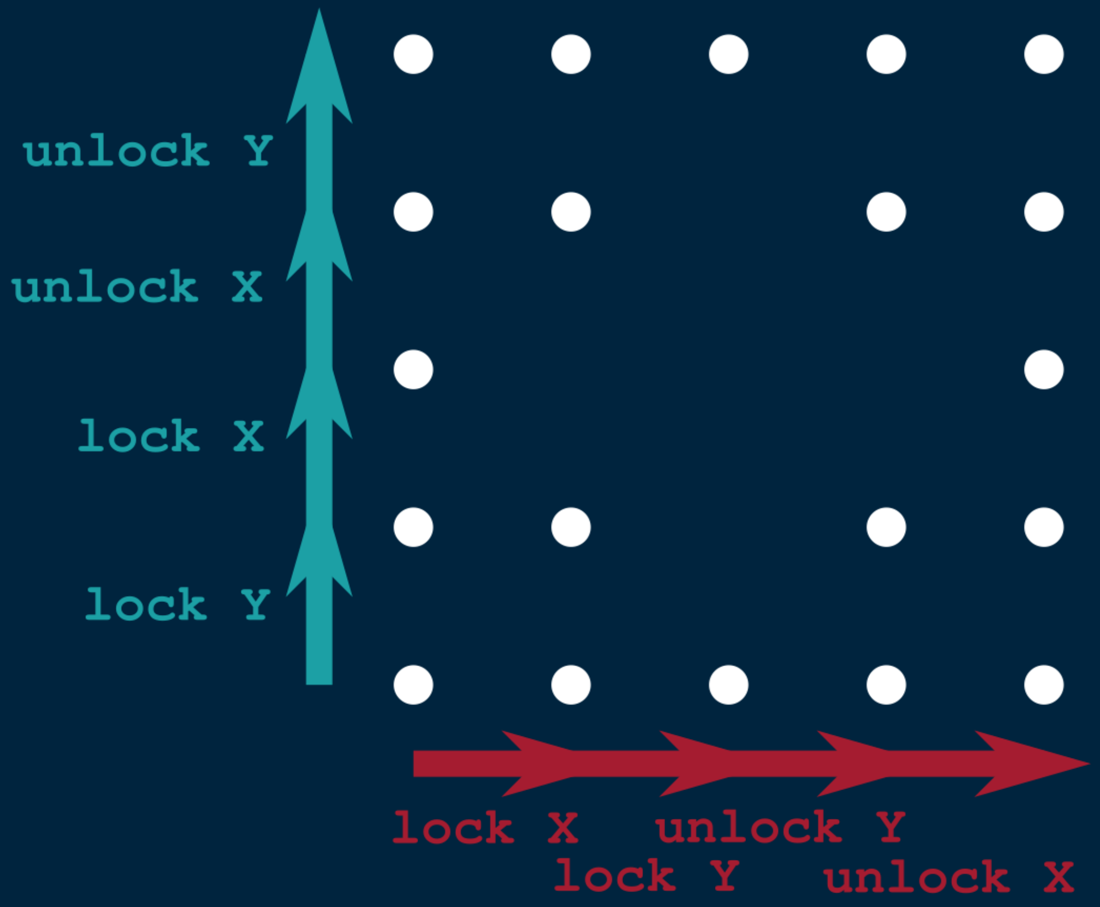
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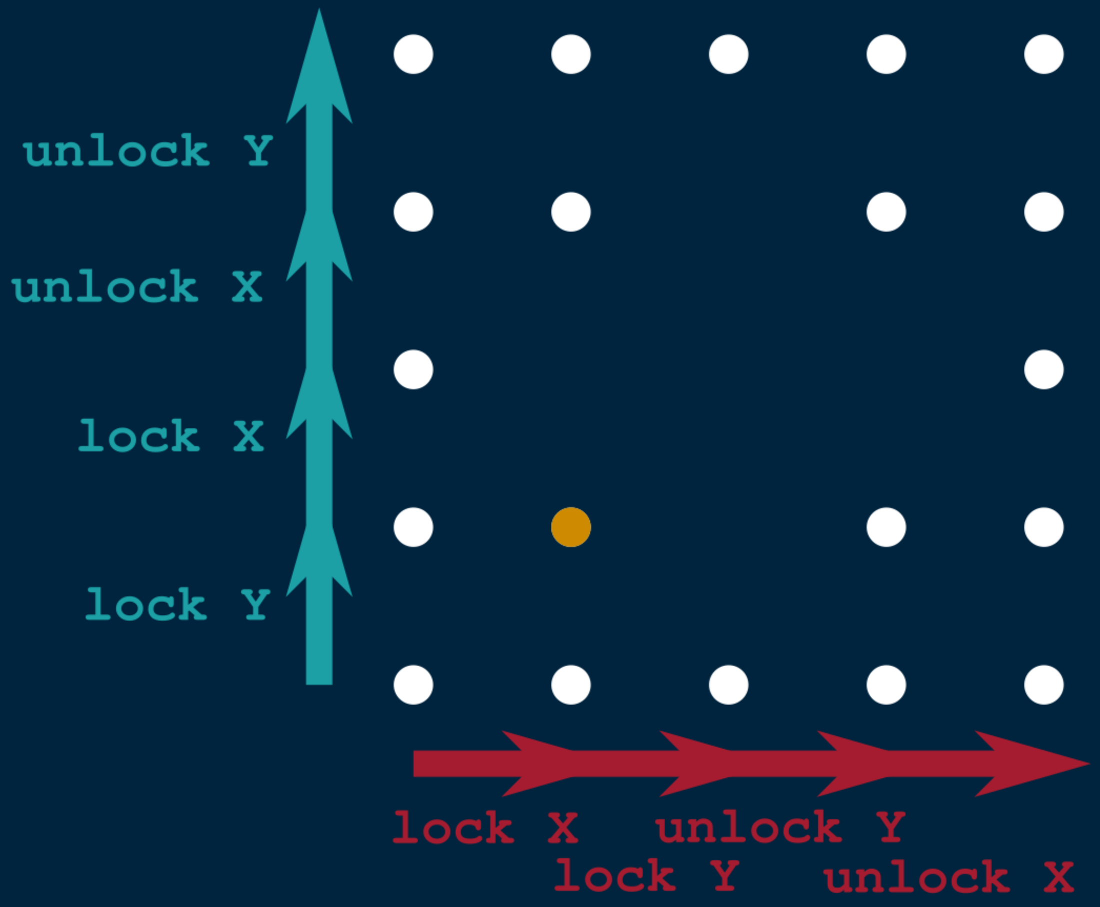
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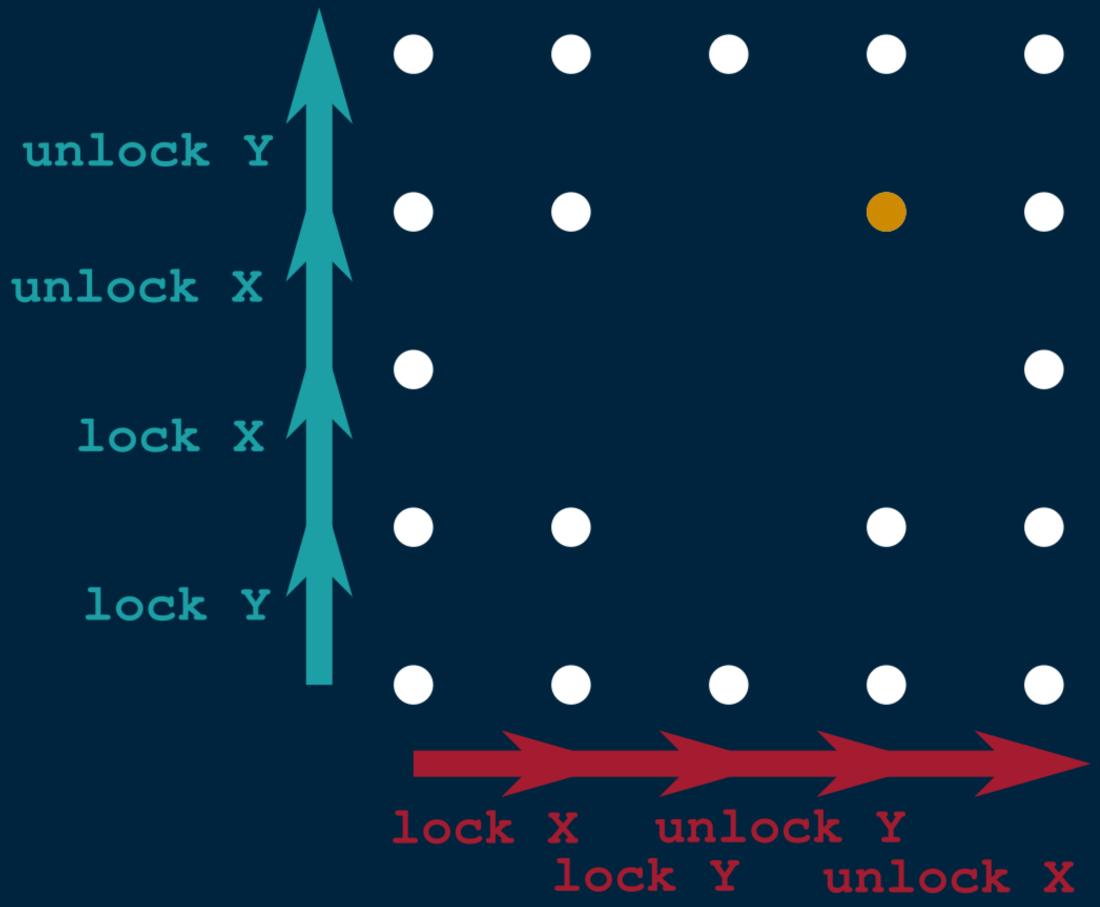
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Interpretation 3: Categories

In directed type theory, each type A comes equipped with the structure of a **category**

$s \qquad\qquad t$

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- Objects are terms $s : A$

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- Objects are terms $s \colon A$
- Morphisms $s \to t$ are terms of type $\mathrm{Hom}(s, t)$

$$\begin{tikzcd} s \arrow[r, bend left, "p"] & t \end{tikzcd}$$

In directed type theory, each type A comes equipped with the structure of a **category**

- Objects are terms $s : A$
- Morphisms $s \to t$ are terms of type $\mathrm{Hom}(s, t)$
- Reflexivity: identity morphisms

$$\begin{tikzcd} s \arrow[loop left] & & t \arrow[loop right] \end{tikzcd}$$

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$$s \qquad\qquad\qquad t$$

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$$s \xrightarrow{p} t \xrightarrow{q} u$$

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- Transitivity: composition of morphisms

$$\begin{tikzcd} s \arrow[r, bend left, "p"] \arrow[rr, bend right, "q \circ p"'] & t \arrow[r, bend left, "q"] & u \end{tikzcd}$$

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\begin{tikzcd}
s \arrow[r, bend left, "p"] & t \arrow[l, bend left, dashed, red]
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Likewise, any function $A \to B$ is a **synthetic functor**: it automatically has a morphism part

$$\frac{F \colon A \to B \qquad p \colon \mathrm{Hom}(s, t)}{\mathrm{map}_F\ p \colon \mathrm{Hom}(F(s), F(t))}$$

which preserves identity morphisms and composition.

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If we're interested in (1,1)-category theory, then we can assert that hom-types between two homs are invertible, i.e. are identity types.

1 Designing a directed type theory

How identity types are defined

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$$\frac{\begin{array}{c}\Gamma \vdash s : A \\ \Gamma, x : A, v : \mathsf{Id}(s, x) \vdash M(x, v) \ \textbf{type} \\ \Gamma \vdash m : M(s, \mathsf{refl}_s)\end{array}}{\Gamma, x : A, v : \mathsf{Id}(s, x) \vdash \mathsf{J}\ m\ (x, v) : M(x, v)}$$

Properties of equality

- Reflexivity asserted axiomatically:

$$\frac{t : A}{\mathsf{refl}_t : \mathsf{Id}(t, t)}$$

- Symmetry can be proved:

$$\frac{p : \mathsf{Id}(s, t)}{p^{-1} : \mathsf{Id}(t, s)}$$

- Transitivity can be proved:

$$\frac{p : \mathsf{Id}(s, t) \quad q : \mathsf{Id}(t, u)}{p \cdot q : \mathsf{Id}(s, u)}$$

- Congruence can be proved:

$$\frac{x : A \vdash B(x)\ \mathbf{type} \quad p : \mathsf{Id}(s, t) \quad b : B(s)}{\mathsf{tr}_B\ p\ b : B(t)}$$

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Difficulty: How do we make transitivity provable, but not symmetry?

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Annotating polarities

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- Encodes the proposition that s *becomes* t

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Solves the issue: we know that $p\colon \mathrm{Hom}(s,t)$ can't be turned into $p^{-1}\colon \mathrm{Hom}(t,s)$, because the type $\mathrm{Hom}(t,s)$ doesn't even make sense!

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Coercions between polarities

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How hom types are defined

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Coercion between polarities is only allowed if the term doesn’t depend on polarized variables

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Coercions between polarities

Want to allow a term s to be considered as either a term of type A or A^-, but only if the polarized context zone is empty

$$\frac{\Gamma \mid \bullet \vdash s \colon A^-}{\Gamma \mid \bullet \vdash -s \colon A} \qquad \frac{\Gamma \mid \bullet \vdash t \colon A}{\Gamma \mid \bullet \vdash -t \colon A^-} \qquad - -t = t$$

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Other topics

- Semantics

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- Synthetic category theory

Thank you!