



GATs, Cats, and CwFs

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What is this talk?

- Background for my forthcoming PhD thesis [Neu25, Chapter 1]
- Interpretation of [KKA19]
 - ▶ Restrict their signature language for QIITs to get a signature language for GATs
 - ▶ Matches the notion of GAT given by [Car86], except we don't allow for equations between sort. This provides an intrinsic presentation of GATs.
- Slides
 - ▶ jacobneu.com/WG6
 - ▶ jacobneu.com/WG6-extended

GAT signature

Category of Algs.

Concrete CwF

Categories

CwFs

GAT signatures



Categories of GAT Algebras

```
def Cat : GAT := {  
  Obj   : U,  
  Hom   : Obj  $\Rightarrow$  Obj  $\Rightarrow$  U,  
  id    : (I : Obj)  $\Rightarrow$  Hom I I,  
  comp  : {I J K : Obj}  $\Rightarrow$   
    Hom J K  $\Rightarrow$  Hom I J  $\Rightarrow$  Hom I K,
```

$\text{lunit} : \{I\ J : \text{Obj}\} \Rightarrow (j : \text{Hom}\ I\ J) \Rightarrow$

$\text{comp}\ (\text{id}\ J)\ j \equiv j,$

$\text{runit} : \{I\ J : \text{Obj}\} \Rightarrow (j : \text{Hom}\ I\ J) \Rightarrow$

$\text{comp}\ j\ (\text{id}\ I) \equiv j,$

$\text{assoc} : \{I\ J\ K\ L : \text{Obj}\} \Rightarrow (j : \text{Hom}\ I\ J) \Rightarrow$

$(k : \text{Hom}\ J\ K) \Rightarrow (\ell : \text{Hom}\ K\ L) \Rightarrow$

$\text{comp}\ \ell\ (\text{comp}\ k\ j) \equiv \text{comp}\ (\text{comp}\ \ell\ k)\ j$

}

GAT signature

Category of Algs.

Categories

$\mathcal{C}at$

Cat

```
def Cwf : GAT := {  
  include Cat as (Con, Sub, comp, id, _, _, _);  
  empty    : Con,  
   $\epsilon$       : ( $\Gamma$  : Con)  $\Rightarrow$  Sub  $\Gamma$  empty,  
   $\epsilon_{\eta}$     : ( $\Gamma$  : Con)  $\Rightarrow$  (f : Sub  $\Gamma$  empty)  $\Rightarrow$   
    f  $\equiv$  ( $\epsilon$   $\Gamma$ ),
```

$$\begin{aligned} \text{Ty} & : \text{Con} \Rightarrow \mathbf{U}, \\ \text{substTy} & : \{\Delta \Gamma : \text{Con}\} \Rightarrow \\ & \quad \text{Sub } \Delta \Gamma \Rightarrow \text{Ty } \Gamma \Rightarrow \text{Ty } \Delta, \\ \text{idTy} & : \{\Gamma : \text{Con}\} \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow \\ & \quad \text{substTy } (\text{id } \Gamma) A \equiv A, \\ \text{compTy} & : \{\Theta \Delta \Gamma : \text{Con}\} \Rightarrow (A : \text{Ty } \Gamma) \\ & \quad (\delta : \text{Sub } \Theta \Delta) \Rightarrow (\gamma : \text{Sub } \Delta \Gamma) \Rightarrow \\ & \quad \text{substTy } \gamma (\text{substTy } \delta A) \\ & \quad \equiv \text{substTy } (\text{comp } \gamma \delta) A, \end{aligned}$$

$$\begin{aligned} \text{Tm} & : (\Gamma : \text{Con}) \Rightarrow \text{Ty } \Gamma \Rightarrow \mathbf{U}, \\ \text{substTm} & : \{\Delta \Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow \\ & (\gamma : \text{Sub } \Delta \Gamma) \Rightarrow \\ & \text{Tm } \Gamma A \Rightarrow \text{Tm } \Delta (\text{substTy } \gamma A), \end{aligned}$$

$$\begin{aligned} \text{idTm} &: \{\Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow (\text{t} : \text{Tm } \Gamma \ A) \\ &\quad \text{substTm } (\text{id } \Gamma) \ \text{t} \quad \# \langle \text{idTy } A \rangle \\ &\equiv \text{t}, \end{aligned}$$

$$\begin{aligned} \text{compTm} &: \{\Theta \ \Delta \ \Gamma : \text{Con}\} \Rightarrow \\ &\quad \{A : \text{Ty } \Gamma\} \Rightarrow (\text{t} : \text{Tm } \Gamma \ A) \Rightarrow \\ &\quad (\delta : \text{Sub } \Theta \ \Delta) \Rightarrow (\gamma : \text{Sub } \Delta \ \Gamma) \Rightarrow \\ &\quad \text{substTm } \gamma \ (\text{substTm } \delta \ \text{t}) \\ &\quad \# \langle \text{compTy } A \ \gamma \ \delta \rangle \\ &\equiv \text{substTm } (\text{comp } \gamma \ \delta) \ \text{t}, \end{aligned}$$

$$\begin{aligned} \text{ext} & : (\Gamma : \text{Con}) \Rightarrow \text{Ty } \Gamma \Rightarrow \text{Con}, \\ \text{pair} & : \{\Delta \Gamma : \text{Con}\} \Rightarrow \{A : \text{Ty } \Gamma\} \Rightarrow \\ & (\gamma : \text{Sub } \Delta \Gamma) \Rightarrow \\ & \text{Tm } \Delta (\text{substTy } \gamma A) \Rightarrow \\ & \text{Sub } \Delta (\text{ext } \Gamma A), \end{aligned}$$

$$\begin{aligned} \text{pair_nat}: \{ \Theta \ \Delta \ \Gamma : \text{Con} \} &\Rightarrow \{ A : \text{Ty} \ \Gamma \} \Rightarrow \\ &(\gamma : \text{Sub} \ \Delta \ \Gamma) \Rightarrow \\ &(\mathfrak{t} : \text{Tm} \ \Delta \ (\text{substTy} \ \gamma \ A)) \Rightarrow \\ &(\delta : \text{Sub} \ \Theta \ \Delta) \Rightarrow \\ &\text{comp} \ (\text{pair} \ \gamma \ \mathfrak{t}) \ \delta \\ &\equiv \text{pair} \ (\text{comp} \ \gamma \ \delta) \\ &\quad (\text{substTm} \ \delta \ \mathfrak{t} \ \# \langle \text{compTy} \ A \ \gamma \ \delta \rangle), \end{aligned}$$

$$\begin{aligned} p & : \{\Gamma : \mathsf{Con}\} \Rightarrow (A : \mathsf{Ty} \ \Gamma) \Rightarrow \\ & \quad \mathsf{Sub} \ (\mathsf{ext} \ \Gamma \ A) \ \Gamma \\ v & : \{\Gamma : \mathsf{Con}\} \Rightarrow (A : \mathsf{Ty} \ \Gamma) \Rightarrow \\ & \quad \mathsf{Tm} \ (\mathsf{ext} \ \Gamma \ A) \ (\mathsf{substTy} \ (p \ A) \ A), \end{aligned}$$

Categories with Families

$$\begin{aligned} \text{ext}_\beta & : (\Delta \Gamma : \text{Con}) \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow \\ & (\gamma : \text{Sub } \Delta \Gamma) \Rightarrow \\ & (t : \text{Tm } \Delta (\text{substTy } \gamma A)) \Rightarrow \\ & \text{comp } (p A) (\text{pair } \gamma t) \equiv \gamma, \\ \text{ext}_\beta & : (\Delta \Gamma : \text{Con}) \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow \\ & (\gamma : \text{Sub } \Delta \Gamma) \Rightarrow \\ & (t : \text{Tm } \Delta (\text{substTy } \gamma A)) \Rightarrow \\ & \text{substTm } (\text{pair } \gamma t) (v A) \\ & \quad \# \langle \text{compTy } A (p A) (\text{pair } \gamma t) \rangle \\ & \quad \# \langle \text{ext}_\beta \gamma t \rangle \\ & \equiv t, \\ \text{ext}_\eta & : (\Gamma : \text{Con}) \Rightarrow (A : \text{Ty } \Gamma) \Rightarrow \\ & \text{pair } (p A) (v A) \\ & \equiv \text{id } (\text{ext } \Gamma A) \end{aligned}$$

GAT signature

Category of Algs.

Categories

$\mathcal{Cat}$

$\mathbf{Cat}$

CwFs

$\mathcal{CwF}$

$\mathbf{CwF}$



```

def CwF+ : GAT := {
  include CwF;
  V : {Γ : Con} ⇒ Ty Γ,
  E1 : {Γ : Con} ⇒ Tm Γ V ⇒ Ty Γ,
  ...
  Eq : {Γ : Con} ⇒ {A : Ty Γ} ⇒
    (t t' : Tm Γ A) ⇒ Ty Γ,
  ...
  Pi : {Γ : Con} ⇒ (X : Tm Γ V) ⇒
    Ty (ext Γ (E1 X)) ⇒ Ty Γ,

```

**Why this specific
GAT?**



The GAT Signature Language

The oneGAT language

We assume that the GAT $\mathcal{CwF} + V + Eq + \Pi_{sd}$ has an initial algebra—the language we'll call **ONEGAT**—given as a QIIT in the metatheory.

$$\begin{aligned} \text{GAT} &:= \overline{\text{Con}} : \text{Set} \\ \overline{\text{Sub}} &: \overline{\text{Con}} \rightarrow \overline{\text{Con}} \rightarrow \text{Set} \\ \overline{\text{Ty}} &: \overline{\text{Con}} \rightarrow \text{Set} \\ \overline{\text{Tm}} &: (\mathcal{G} : \overline{\text{Con}}) \rightarrow \overline{\text{Ty}} \mathcal{G} \end{aligned}$$

Use the DSL parsing utilities in `LEAN4` to translate the intuitive syntax for GATs into **ONEGAT**.

def $\mathfrak{N} : \text{GAT} := \{$	$\diamond$
$\text{Nat} : \mathbf{U},$	$\triangleright V$
$\text{zero} : \text{Nat},$	$\triangleright \text{El } 0$
$\text{succ} : \text{Nat} \Rightarrow \text{Nat}$	$\triangleright \prod 1 (\text{El } 2)$
$\}$	

Categories

```
def Cat : GAT := {  
  Obj   : U,  
  Hom   : Obj  $\Rightarrow$  Obj  $\Rightarrow$  U,  
  id    : (I : Obj)  $\Rightarrow$  Hom I I,  
  comp  : {I J K : Obj}  $\Rightarrow$   
    Hom J K  $\Rightarrow$  Hom I J  $\Rightarrow$  Hom I K,  
  lunit : {I J : Obj}  $\Rightarrow$  (j : Hom I J)  $\Rightarrow$   
    comp (id J) j  $\equiv$  j,  
  runit : {I J : Obj}  $\Rightarrow$  (j : Hom I J)  $\Rightarrow$   
    comp j (id I)  $\equiv$  j,  
  assoc : {I J K L : Obj}  $\Rightarrow$   
    (j : Hom I J)  $\Rightarrow$  (k : Hom J K)  $\Rightarrow$   
    (l : Hom K L)  $\Rightarrow$   
    comp l (comp k j)  
     $\equiv$  comp (comp l k) j  
}
```

◇

- ▷ V
- ▷ $\prod 0 (\prod 1 V)$
- ▷ $\prod 1 (El (1 @ 0 @ 0))$
- ▷ $\prod 2 (\prod 3 (\prod 4 (\prod (4 @ 1 @ 0) (\prod (5 @ 3 @ 2) (El (6 @ 4 @ 2))))))$
- ▷ $\prod 3 (\prod 4 (\prod (4 @ 1 @ 0) (Eq (3 @ 2 @ 1 @ 1 @ (4 @ 1) @ 0) 0)))$
- ▷ $\prod 4 (\prod 5 (\prod (5 @ 1 @ 0) (Eq (4 @ 2 @ 2 @ 1 @ 0 @ (5 @ 2)) 0)))$
- ▷ $\prod 5 (\prod 6 (\prod 7 (\prod 8 (\prod (8 @ 3 @ 2) (\prod (9 @ 3 @ 2) (\prod (10 @ 3 @ 2) (Eq (9 @ 6 @ 5 @ 3 @ 0 @ (9 @ 6 @ 5 @ 4 @ 1 @ 2)) (9 @ 6 @ 4 @ 3 @ (9 @ 5 @ 4 @ 3 @ 0 @ 1) @ 2))))))))$

$\mathcal{C}at$

- ▷ $El\ 6$
- ▷ $\prod 7\ (El\ (7\ @\ 0\ @\ 1))$
- ▷ $\prod 8\ (\prod (8\ @\ 0\ @\ 2)\ (Eq\ 0\ (2\ @\ 1)))$
- ▷ $\prod 9\ V$
- ▷ $\prod 10\ (\prod 11\ (\prod (11\ @\ 1\ @\ 0)\ (\prod (3\ @\ 1)\ (El\ (4\ @\ 3)))))$
- ▷ $\prod 11\ (\prod (2\ @\ 0)\ (Eq\ (2\ @\ 1\ @\ 1\ @\ (11\ @\ 1)\ @\ 0)\ 0))$
- ▷ $\prod 12\ (\prod 13\ (\prod 14\ (\prod (5\ @\ 0)\ (\prod (15\ @\ 3\ @\ 2)\ (\prod (16\ @\ 3\ @\ 2)\ (Eq\ (7\ @\ 4\ @\ 3\ @\ 0\ @\ (7\ @\ 5\ @\ 4\ @\ 1\ @\ 2))\ (7\ @\ 5\ @\ 3\ @\ (15\ @\ 5\ @\ 4\ @\ 3\ @\ 0\ @\ 1)\ @\ 2))))))$
- ▷ $\prod 13\ (\prod (4\ @\ 0)\ V)$
- ▷ $\prod 14\ (\prod 15\ (\prod (6\ @\ 0)\ (\prod (16\ @\ 2\ @\ 1)\ (\prod (4\ @\ 2\ @\ 1)\ (El\ (5\ @\ 4\ @\ (8\ @\ 4\ @\ 3\ @\ 1\ @\ 2))))))$
- ▷ $\prod 15\ (\prod (6\ @\ 0)\ (\prod (3\ @\ 1\ @\ 0)\ (Eq\ (transp\ (6\ @\ 2\ @\ 1)\ (3\ @\ 2\ @\ 2\ @\ 1\ @\ (16\ @\ 2)\ @\ 0))\ 0)))$
- ▷ $\prod 16\ (\prod 17\ (\prod 18\ (\prod (9\ @\ 0)\ (\prod (6\ @\ 1\ @\ 0)\ (\prod (20\ @\ 4\ @\ 3)\ (\prod (21\ @\ 4\ @\ 3)\ (Eq\ (transp\ (10\ @\ 6\ @\ 5\ @\ 4\ @\ 3\ @\ 0\ @\ 1)\ (8\ @\ 5\ @\ 4\ @\ 3\ @\ 0\ @\ (8\ @\ 6\ @\ 5\ @\ (12\ @\ 5\ @\ 4\ @\ 0\ @\ 2)\ @\ 1\ @\ 2))\ (8\ @\ 6\ @\ 4\ @\ 3\ @\ (20\ @\ 6\ @\ 5\ @\ 4\ @\ 0\ @\ 1)\ @\ 2))))))$

- ▷ $\Pi 17 (\Pi (8 @ 0) (EI 19))$
- ▷ $\Pi 18 (\Pi 19 (\Pi (10 @ 0) (\Pi (20 @ 2 @ 1) (\Pi (8 @ 3 @ (11 @ 3 @ 2 @ 0 @ 1)) (EI (22 @ 4 @ (5 @ 3 @ 2)))))))$
- ▷ $\Pi 19 (\Pi 20 (\Pi 21 (\Pi (12 @ 0) (\Pi (22 @ 2 @ 1) (\Pi (10 @ 3 @ (13 @ 3 @ 2 @ 0 @ 1)) (\Pi (24 @ 5 @ 4) (Eq (23 @ 6 @ 5 @ (8 @ 4 @ 3) @ (7 @ 5 @ 4 @ 3 @ 2 @ 1) @ 0) (7 @ 6 @ 4 @ 3 @ (23 @ 6 @ 5 @ 4 @ 2 @ 0) @ (transp (13 @ 6 @ 5 @ 4 @ 3 @ 2 @ 0) (11 @ 6 @ 5 @ (15 @ 5 @ 4 @ 2 @ 3) @ 0 @ 1))))))))))$
- ▷ $\Pi 20 (\Pi (11 @ 0) (EI (21 @ (4 @ 1 @ 0) @ 1)))$
- ▷ $\Pi 21 (\Pi (12 @ 0) (EI (9 @ (5 @ 1 @ 0) @ (12 @ (5 @ 1 @ 0) @ 1 @ (2 @ 1 @ 0) @ 0))))$
- ▷ $\Pi 22 (\Pi 23 (\Pi (14 @ 0) (\Pi (24 @ 2 @ 1) (\Pi (12 @ 3 @ (15 @ 3 @ 2 @ 0 @ 1)) (Eq (24 @ 4 @ (9 @ 3 @ 2) @ 3 @ (6 @ 3 @ 2) @ (8 @ 4 @ 3 @ 2 @ 1 @ 0) 1))))))$
- ▷ $\Pi 23 (\Pi 24 (\Pi (15 @ 0) (\Pi (25 @ 2 @ 1) (\Pi (13 @ 3 @ (16 @ 3 @ 2 @ 0 @ 1)) (Eq (transp (5 @ 4 @ 3 @ 2 @ 1 @ 0) (transp (15 @ 4 @ (10 @ 3 @ 2) @ 3 @ 2 @ (7 @ 3 @ 2) @ (9 @ 4 @ 3 @ 2 @ 1 @ 0)) (13 @ 4 @ (10 @ 3 @ 2) @ (17 @ 4 @ 3 @ 1 @ 2) @ (9 @ 4 @ 3 @ 2 @ 1 @ 0) @ (6 @ 3 @ 2)))) 0))))))$
- ▷ $\Pi 24 (\Pi (15 @ 0) (Eq (7 @ (8 @ 1 @ 0) @ 1 @ 0 @ (5 @ 1 @ 0) @ (4 @ 1 @ 0)) (24 @ (8 @ 1 @ 0))))$

To the logical
extreme: oneGAT
describes itself

Quotient Inductive-Inductive definitions over all GATs

The benefit of quotient inductive-inductive types is that we can reason about them by quotient induction induction.

Given an appropriate *motive* $M_{\overline{\text{Con}}}$, $M_{\overline{\text{Sub}}}$, $M_{\overline{\text{T}_y}}$, $M_{\overline{\text{T}_m}}$ and appropriate *method* m , we get:

$$(\text{elim } m)_{\overline{\text{Con}}}: (\mathfrak{G}: \text{GAT}) \rightarrow M_{\overline{\text{Con}}}(\mathfrak{G})$$

$$(\text{elim } m)_{\overline{\text{Sub}}}: (\mathfrak{G} \mathfrak{H}: \overline{\text{Con}}) \rightarrow (\mathfrak{s}: \overline{\text{Sub}} \mathfrak{G} \mathfrak{H}) \rightarrow \\ M_{\overline{\text{Sub}}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{Con}}} \mathfrak{H}, \mathfrak{s})$$

$$(\text{elim } m)_{\overline{\text{T}_y}}: (\mathfrak{G}: \overline{\text{Con}}) \rightarrow (\mathcal{X}: \overline{\text{T}_y} \mathfrak{G}) \rightarrow M_{\overline{\text{T}_y}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, \mathcal{X})$$

$$(\text{elim } m)_{\overline{\text{T}_m}}: (\mathfrak{G}: \overline{\text{Con}}) \rightarrow (\mathcal{X}: \overline{\text{T}_y} \mathfrak{G}) \rightarrow (X: \overline{\text{T}_m}(\mathfrak{G}, \mathcal{X})) \\ \rightarrow M_{\overline{\text{T}_m}}((\text{elim } m)_{\overline{\text{Con}}} \mathfrak{G}, (\text{elim } m)_{\overline{\text{T}_y}} \mathcal{X}, X)$$

$_ - \text{Alg} : (\mathfrak{G} : \text{GAT}) \rightarrow \text{Set}$

$\text{Hom} : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

From [KKA19, Appendix A]

Syntax	Algebras		
$\Gamma : \text{Con}$	$\Gamma^\Delta : \text{Set}$	$\triangleright \eta : (\pi_1 \sigma, \pi_2 \sigma) = \sigma$	$\triangleright \eta^\Delta : \equiv \text{refl}$
$A : \text{Ty } \Gamma$	$A^\Delta : \Gamma^\Delta \rightarrow \text{Set}$	$, \circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$, \circ^\Delta : \equiv \text{refl}$
$\sigma : \text{Sub } \Gamma \Delta$	$\sigma^\Delta : \Gamma^\Delta \rightarrow \Delta^\Delta$	$\text{U} : \text{Ty } \Gamma$	$\text{U}^\Delta \gamma : \equiv \text{Set}$
$t : \text{Tm } \Gamma A$	$t^\Delta : (\gamma : \Gamma^\Delta) \rightarrow A^\Delta \gamma$	$\text{El } (a : \text{Tm } \Gamma \text{U}) : \text{Ty } \Gamma$	$(\text{El } a)^\Delta \gamma : \equiv a^\Delta \gamma$
$\cdot : \text{Con}$	$\cdot^\Delta : \equiv \top$	$\text{U} [] : \text{U} [\sigma] = \text{U}$	$\text{U} []^\Delta : \equiv \text{refl}$
$\Gamma \triangleright A : \text{Con}$	$(\Gamma \triangleright A)^\Delta : \equiv (\gamma : \Gamma^\Delta) \times A^\Delta \gamma$	$\text{El} [] : (\text{El } a) [\sigma] = \text{El } (a[\sigma])$	$\text{El} []^\Delta : \equiv \text{refl}$
$(A : \text{Ty } \Delta) [\sigma : \text{Sub } \Gamma \Delta] : \text{Ty } \Gamma$	$(A[\sigma])^\Delta \gamma : \equiv A^\Delta (\sigma^\Delta \gamma)$	$\Pi (a : \text{Tm } \Gamma \text{U}) (B : \text{Ty } (\Gamma \triangleright \text{El } a)) : \text{Ty } \Gamma$	$(\Pi a B)^\Delta \gamma : \equiv (\alpha : a^\Delta \gamma) \rightarrow B^\Delta (\gamma, \alpha)$
$\text{id} : \text{Sub } \Gamma \Gamma$	$\text{id}^\Delta \gamma : \equiv \gamma$	$\text{app } (t : \text{Tm } \Gamma (\Pi a B)) : \text{Tm } (\Gamma \triangleright \text{El } a) B$	$(\text{app } t)^\Delta (\gamma, \alpha) : \equiv t^\Delta \gamma \alpha$
$(\sigma : \text{Sub } \Theta \Delta) \circ (\delta : \text{Sub } \Gamma \Theta) : \text{Sub } \Gamma \Delta$	$(\sigma \circ \delta)^\Delta \gamma : \equiv \sigma^\Delta (\delta^\Delta \gamma)$	$\Pi [] : (\Pi a B) [\sigma] = \Pi (a[\sigma]) (B[\sigma^\uparrow])$	$\Pi []^\Delta : \equiv \text{refl}$
$\epsilon : \text{Sub } \Gamma \cdot$	$\epsilon^\Delta \gamma : \equiv \text{tt}$	$\text{app} [] : (\text{app } t) [\sigma^\uparrow] = \text{app } (t[\sigma])$	$\text{app} []^\Delta : \equiv \text{refl}$
$(\sigma : \text{Sub } \Gamma \Delta), (t : \text{Tm } \Gamma (A[\sigma])) : \text{Sub } \Gamma (\Delta \triangleright A)$	$(\sigma, t)^\Delta \gamma : \equiv (\sigma^\Delta \gamma, t^\Delta \gamma)$	$\text{Id } (a : \text{Tm } \Gamma \text{U}) (t u : \text{Tm } \Gamma (\text{El } a)) : \text{Ty } \Gamma$	$(\text{Id } a t u)^\Delta \gamma : \equiv (t^\Delta \gamma = u^\Delta \gamma)$
$\pi_1 (\sigma : \text{Sub } \Gamma (\Delta \triangleright A)) : \text{Sub } \Gamma \Delta$	$(\pi_1 \sigma)^\Delta \gamma : \equiv \text{proj}_1 (\sigma^\Delta \gamma)$	$\text{reflect } (e : \text{Tm } \Gamma (\text{Id } a t u)) : t = u$	$(\text{reflect } e)^\Delta : \equiv \text{funext } e^\Delta$
$\pi_2 (\sigma : \text{Sub } \Gamma (\Delta \triangleright A)) : \text{Tm } \Gamma (A[\pi_1 \sigma])$	$(\pi_2 \sigma)^\Delta \gamma : \equiv \text{proj}_2 (\sigma^\Delta \gamma)$	$\text{Id} [] : (\text{Id } a t u) [\sigma] = \text{Id } (a[\sigma]) (t[\sigma]) (u[\sigma])$	$\text{Id} []^\Delta : \equiv \text{refl}$
$(t : \text{Tm } \Delta A) [\sigma : \text{Sub } \Gamma \Delta] : \text{Tm } \Gamma (A[\sigma])$	$(t[\sigma])^\Delta \gamma : \equiv t^\Delta (\sigma^\Delta \gamma)$	$\hat{\Pi} (T : \text{Set}) (B : T \rightarrow \text{Ty } \Gamma) : \text{Ty } \Gamma$	$(\hat{\Pi} T B)^\Delta \gamma : \equiv (\alpha : T) \rightarrow (B \alpha)^\Delta \gamma$
$[\text{id}] : A[\text{id}] = A$	$[\text{id}]^\Delta : \equiv \text{refl}$	$(t : \text{Tm } \Gamma (\hat{\Pi} T B)) \hat{\otimes} (\alpha : T) : \text{Tm } \Gamma (B \alpha)$	$(t \hat{\otimes} \alpha)^\Delta \gamma : \equiv t^\Delta \gamma \alpha$
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^\Delta : \equiv \text{refl}$	$\hat{\Pi} [] : (\hat{\Pi} T B) [\sigma] = \hat{\Pi} T (\lambda \alpha. (B \alpha) [\sigma])$	$\hat{\Pi} []^\Delta : \equiv \text{refl}$
$\text{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\text{ass}^\Delta : \equiv \text{refl}$	$\hat{\otimes} [] : (t \hat{\otimes} \alpha) [\sigma] = (t[\sigma]) \hat{\otimes} \alpha$	$\hat{\otimes} []^\Delta : \equiv \text{refl}$
$\text{idl} : \text{id} \circ \sigma = \sigma$	$\text{idl}^\Delta : \equiv \text{refl}$		
$\text{idr} : \sigma \circ \text{id} = \sigma$	$\text{idr}^\Delta : \equiv \text{refl}$		
$\cdot \eta : \{\sigma : \text{Sub } \Gamma \cdot\} \rightarrow \sigma = \epsilon$	$\cdot \eta^\Delta : \equiv \text{refl}$		
$\triangleright \beta_1 : \pi_1 (\sigma, t) = \sigma$	$\triangleright \beta_1^\Delta : \equiv \text{refl}$		
$\triangleright \beta_2 : \pi_2 (\sigma, t) = t$	$\triangleright \beta_2^\Delta : \equiv \text{refl}$		

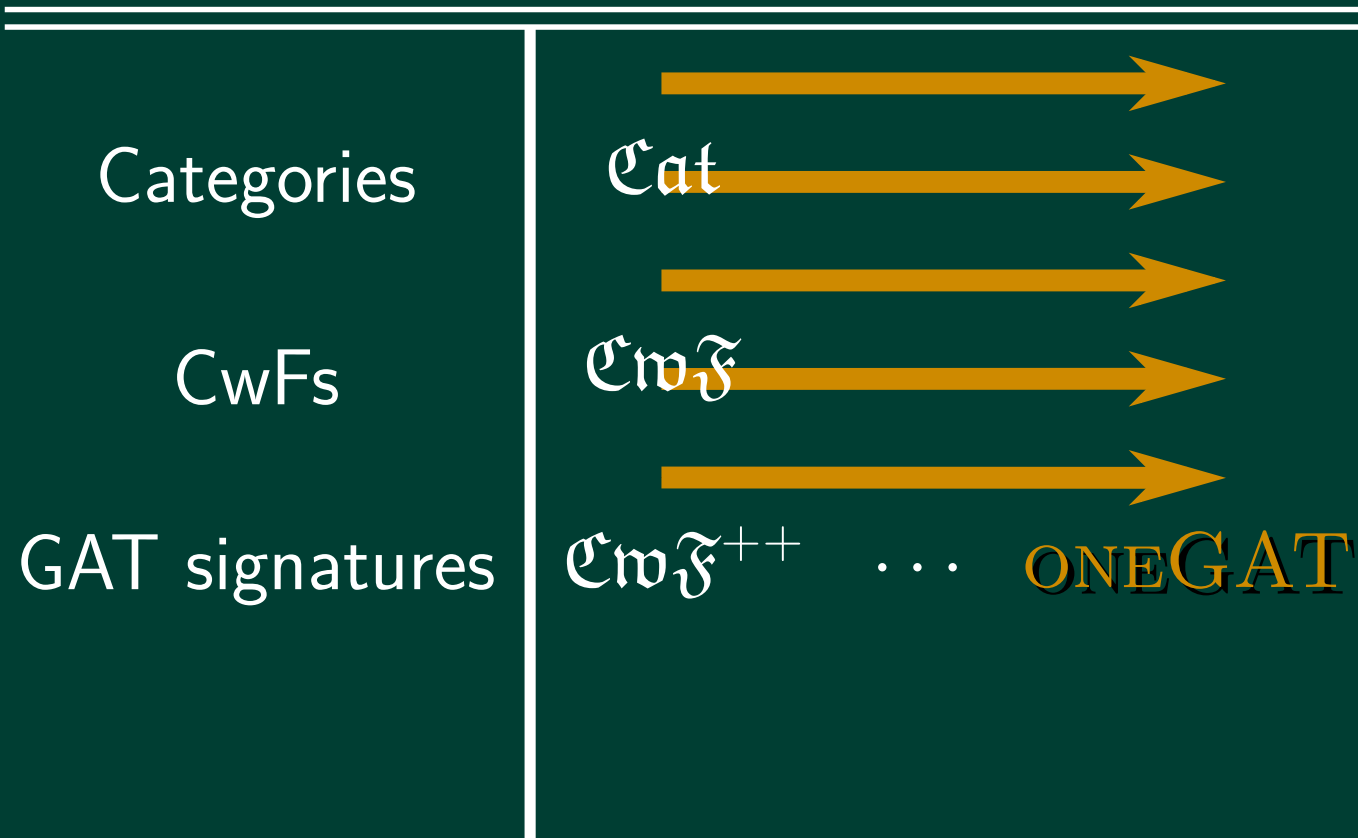
$$(_)^\Delta : (\mathfrak{G} : \text{GAT}) \rightarrow \text{Set}$$

From [KKA19, Appendix A]

$\Gamma : \text{Con}$	$\Gamma^{\mathbf{M}} : \Gamma^{\mathbf{A}} \rightarrow \Gamma^{\mathbf{A}} \rightarrow \text{Set}$	$\text{El}(a : \text{Tm } \Gamma \text{ U}) : \text{Ty } \Gamma$	$(\text{El } a)^{\mathbf{M}} \gamma^{\mathbf{M}} \alpha^0 \alpha^I \equiv a^{\mathbf{M}} \gamma^{\mathbf{M}} \alpha^0 = \alpha^I$
$A : \text{Ty } \Gamma$	$A^{\mathbf{M}} : \Gamma^{\mathbf{M}} \gamma^0 \gamma^I \rightarrow A^{\mathbf{A}} \gamma^0 \rightarrow A^{\mathbf{A}} \gamma^I \text{Set}$	$\text{U}[] : \text{U}[\sigma] = \text{U}$	$\text{U}[]^{\mathbf{M}} \equiv \text{refl}$
$\sigma : \text{Sub } \Gamma \Delta$	$\sigma^{\mathbf{M}} : \Gamma^{\mathbf{M}} \gamma^0 \gamma^I \rightarrow \Delta^{\mathbf{M}} (\sigma^{\mathbf{A}} \gamma^0) (\sigma^{\mathbf{A}} \gamma^I)$	$\text{El}[] : (\text{El } a)[\sigma] = \text{El}(a[\sigma])$	$\text{El}[]^{\mathbf{M}} \equiv \text{refl}$
$t : \text{Tm } \Gamma A$	$t^{\mathbf{M}} : (\gamma^{\mathbf{M}} : \Gamma^{\mathbf{M}} \gamma^0 \gamma^I) \rightarrow A^{\mathbf{M}} \gamma^{\mathbf{M}} (t^{\mathbf{A}} \gamma^0) (t^{\mathbf{A}} \gamma^I)$	$\Pi(a : \text{Tm } \Gamma \text{ U})(B : \text{Ty } (\Gamma \triangleright \text{El } a)) : \text{Ty } \Gamma$	$(\Pi a B)^{\mathbf{M}} \gamma^{\mathbf{M}} f^0 f^I \equiv (\alpha^0 : a^{\mathbf{A}} \gamma^0) \rightarrow B^{\mathbf{M}} (\gamma^{\mathbf{M}}, \text{refl}) (f^0 \alpha^0) (f^I (a^{\mathbf{M}} \gamma^{\mathbf{M}} \alpha^0))$
$\cdot : \text{Con}$	$\cdot^{\mathbf{M}} \text{ tt tt} \equiv \text{T}$	$\text{app}(t : \text{Tm } \Gamma (\Pi a B)) : \text{Tm } (\Gamma \triangleright \text{El } a) B$	$(\text{app } t)^{\mathbf{M}} (\gamma^{\mathbf{M}}, \alpha^{\mathbf{M}}) \equiv \text{J}(t^{\mathbf{M}} \gamma^{\mathbf{M}} \alpha^0) \alpha^{\mathbf{M}}$
$\Gamma \triangleright A : \text{Con}$	$(\Gamma \triangleright A)^{\mathbf{M}} (\gamma^0, \alpha^0) (\gamma^I, \alpha^I) \equiv (\gamma^{\mathbf{M}} : \Gamma^{\mathbf{M}} \gamma^0 \gamma^I) \times A^{\mathbf{M}} \gamma^{\mathbf{M}} \alpha^0 \alpha^I$	$\Pi[] : (\Pi a B)[\sigma] = \Pi(a[\sigma])(B[\sigma^{\uparrow}])$	$\Pi[]^{\mathbf{M}} \equiv \text{refl}$
$(A : \text{Ty } \Delta)[\sigma : \text{Sub } \Gamma \Delta] : \text{Ty } \Gamma$	$(A[\sigma])^{\mathbf{M}} \gamma^{\mathbf{M}} \alpha^0 \alpha^I \equiv A^{\mathbf{M}} (\sigma^{\mathbf{M}} \gamma^{\mathbf{M}}) \alpha^0 \alpha^I$	$\text{app}[] : (\text{app } t)[\sigma \uparrow] = \text{app}(t[\sigma])$	$\text{app}[]^{\mathbf{M}} \equiv \text{refl}$
$\text{id} : \text{Sub } \Gamma \Gamma$	$\text{id}^{\mathbf{M}} \gamma^{\mathbf{M}} \equiv \gamma^{\mathbf{M}}$	$\text{ld}(a : \text{Tm } \Gamma \text{ U})(t u : \text{Tm } \Gamma (\text{El } a)) : \text{Ty } \Gamma$	$(\text{ld } a \text{ } t \text{ } u)^{\mathbf{M}} \gamma^{\mathbf{M}} e^0 e^I \equiv \text{T}$
$(\sigma : \text{Sub } \Theta \Delta) \circ (\delta : \text{Sub } \Gamma \Theta) : \text{Sub } \Gamma \Delta$	$(\sigma \circ \delta)^{\mathbf{M}} \gamma^{\mathbf{M}} \equiv \sigma^{\mathbf{M}} (\delta^{\mathbf{M}} \gamma^{\mathbf{M}})$	$\text{reflect}(e : \text{Tm } \Gamma (\text{ld } a \text{ } t \text{ } u)) : t = u$	$(\text{reflect } e)^{\mathbf{M}} \equiv \text{UIP}$
$\epsilon : \text{Sub } \Gamma \cdot$	$\epsilon^{\mathbf{M}} \gamma^{\mathbf{M}} \equiv \text{tt}$	$\text{ld}[] : (\text{ld } a \text{ } t \text{ } u)[\sigma] = \text{ld}(a[\sigma])(t[\sigma])(u[\sigma])$	$\text{ld}[]^{\mathbf{M}} \equiv \text{refl}$
$(\sigma : \text{Sub } \Gamma \Delta), (t : \text{Tm } \Gamma (A[\sigma])) : \text{Sub } \Gamma (\Delta \triangleright A)$	$(\sigma, t)^{\mathbf{M}} \gamma^{\mathbf{M}} \equiv (\sigma^{\mathbf{M}} \gamma^{\mathbf{M}}, t^{\mathbf{M}} \gamma^{\mathbf{M}})$	$\hat{\Pi}(T : \text{Set})(B : T \rightarrow \text{Ty } \Gamma) : \text{Ty } \Gamma$	$(\hat{\Pi} T B)^{\mathbf{M}} \gamma^{\mathbf{M}} f^0 f^I \equiv (\alpha : T) \rightarrow (B \alpha)^{\mathbf{M}} \gamma^{\mathbf{M}} (f^0 \alpha) (f^I \alpha)$
$\pi_1(\sigma : \text{Sub } \Gamma (\Delta \triangleright A)) : \text{Sub } \Gamma \Delta$	$(\pi_1 \sigma)^{\mathbf{M}} \gamma^{\mathbf{M}} \equiv \text{proj}_1(\sigma^{\mathbf{M}} \gamma^{\mathbf{M}})$	$(t : \text{Tm } \Gamma (\hat{\Pi} T B)) \hat{\circ}(\alpha : T) : \text{Tm } \Gamma (B \alpha)$	$(t \hat{\circ} \alpha)^{\mathbf{M}} \gamma^{\mathbf{M}} \equiv t^{\mathbf{M}} \gamma^{\mathbf{M}} \alpha$
$\pi_2(\sigma : \text{Sub } \Gamma (\Delta \triangleright A)) : \text{Tm } \Gamma (A[\pi_1 \sigma])$	$(\pi_2 \sigma)^{\mathbf{M}} \gamma^{\mathbf{M}} \equiv \text{proj}_2(\sigma^{\mathbf{M}} \gamma^{\mathbf{M}})$	$\hat{\Pi}[] : (\hat{\Pi} T B)[\sigma] = \hat{\Pi} T (\lambda \alpha. (B \alpha)[\sigma])$	$\hat{\Pi}[]^{\mathbf{M}} \equiv \text{refl}$
$(t : \text{Tm } \Delta A)[\sigma : \text{Sub } \Gamma \Delta] : \text{Tm } \Gamma (A[\sigma])$	$(t[\sigma])^{\mathbf{M}} \gamma^{\mathbf{M}} \equiv t^{\mathbf{M}} (\sigma^{\mathbf{M}} \gamma^{\mathbf{M}})$	$\hat{\circ}[] : (t \hat{\circ} \alpha)[\sigma] = (t[\sigma]) \hat{\circ} \alpha$	$\hat{\circ}[]^{\mathbf{M}} \equiv \text{refl}$
$[\text{id}] : A[\text{id}] = A$	$[\text{id}]^{\mathbf{M}} \equiv \text{refl}$		
$[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$	$[\circ]^{\mathbf{M}} \equiv \text{refl}$		
$\text{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$	$\text{ass}^{\mathbf{M}} \equiv \text{refl}$		
$\text{idl} : \text{id} \circ \sigma = \sigma$	$\text{idl}^{\mathbf{M}} \equiv \text{refl}$		
$\text{idr} : \sigma \circ \text{id} = \sigma$	$\text{idr}^{\mathbf{M}} \equiv \text{refl}$		
$\cdot \eta : \{\sigma : \text{Sub } \Gamma \cdot\} \rightarrow \sigma = \epsilon$	$\cdot \eta^{\mathbf{M}} \equiv \text{refl}$		
$\triangleright \beta_1 : \pi_1(\sigma, t) = \sigma$	$\triangleright \beta_1^{\mathbf{M}} \equiv \text{refl}$		
$\triangleright \beta_2 : \pi_2(\sigma, t) = t$	$\triangleright \beta_2^{\mathbf{M}} \equiv \text{refl}$		
$\triangleright \eta : (\pi_1 \sigma, \pi_2 \sigma) = \sigma$	$\triangleright \eta^{\mathbf{M}} \equiv \text{refl}$		
$\circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$	$\circ^{\mathbf{M}} \equiv \text{refl}$		
$\text{U} : \text{Ty } \Gamma$	$\text{U}^{\mathbf{M}} \gamma^{\mathbf{M}} T^0 T^I \equiv T^0 \rightarrow T^I$		

$$(_)^{\mathbf{M}} : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$$

Initial Algebra
Category of Algs.
GAT signature



Idea:

Elements of initial algebra are
only those derivable from the
mere concept of the structure

Construction of initial algebras

def $\mathfrak{N} : \text{GAT} := \{ \text{Nat} : \mathbf{U}, \text{zero} : \text{Nat}, \text{succ} : \text{Nat} \Rightarrow \text{Nat} \}$

$\mathbb{N} := \overline{\text{Tm}}(\mathfrak{N}, \text{Nat})$

$\text{Nat} : \mathbf{U}, \text{zero} : \text{Nat}, \text{succ} : \text{Nat} \Rightarrow \text{Nat} \vdash t : \text{Nat}$

Forms a $\mathfrak{N}$ -algebra with zero and $\lambda t \rightarrow \text{succ } t$.

Claim Can do this for arbitrary GAT $\mathfrak{G}$, by quotient induction-induction on ONEGAT .

Upshot: Any GAT
extension of $\mathcal{E}_{\text{no}\mathcal{F}}$
has an initial model



Concrete CwFs

Ahrens and Lumsdaine [AL19] introduce the notion of **displayed categories**

$$\text{Obj}^D : \text{Obj} \rightarrow \text{Set}$$

$$\text{Hom}^D : (I \ J : \text{Obj}) \rightarrow \text{Obj}^D \ I \rightarrow \text{Obj}^D \ J \rightarrow \text{Hom} \ I \ J \rightarrow \text{Set}$$

$$\text{id}^D : (I : \text{Obj}) \rightarrow (I^D : \text{Obj}^D \ I) \rightarrow \text{Hom}^D \ I \ I \ I^D \ I^D \text{id}_I$$

...

Fact: We can define
“displayed
 $\mathcal{G}$ -algebra” for every
GAT $\mathcal{G}$

$_ - \text{Alg} : (\mathfrak{G} : \text{GAT}) \rightarrow \text{Set}$
 $\text{Hom} : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$
 $_ - \text{DAAlg} : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

From [KKA19, Appendix A]

Syntax

$\Gamma : \text{Con}$
 $A : \text{Ty } \Gamma$
 $\sigma : \text{Sub } \Gamma \Delta$
 $t : \text{Tm } \Gamma A$
 $\cdot : \text{Con}$
 $\Gamma \triangleright A : \text{Con}$
 $(A : \text{Ty } \Delta) [\sigma : \text{Sub } \Gamma \Delta] : \text{Ty } \Gamma$
 $\text{id} : \text{Sub } \Gamma \Gamma$
 $(\sigma : \text{Sub } \Theta \Delta) \circ (\delta : \text{Sub } \Gamma \Theta) : \text{Sub } \Gamma \Delta$
 $\epsilon : \text{Sub } \Gamma \cdot$
 $(\sigma : \text{Sub } \Gamma \Delta), (t : \text{Tm } \Gamma (A[\sigma])) : \text{Sub } \Gamma (\Delta \triangleright A)$
 $\pi_1 (\sigma : \text{Sub } \Gamma (\Delta \triangleright A)) : \text{Sub } \Gamma \Delta$
 $\pi_2 (\sigma : \text{Sub } \Gamma (\Delta \triangleright A)) : \text{Tm } \Gamma (A[\pi_1 \sigma])$
 $(t : \text{Tm } \Delta A) [\sigma : \text{Sub } \Gamma \Delta] : \text{Tm } \Gamma (A[\sigma])$
 $[\text{id}] : A[\text{id}] = A$
 $[\circ] : A[\sigma \circ \delta] = A[\sigma][\delta]$
 $\text{ass} : (\sigma \circ \delta) \circ \nu = \sigma \circ (\delta \circ \nu)$
 $\text{idl} : \text{id} \circ \sigma = \sigma$
 $\text{idr} : \sigma \circ \text{id} = \sigma$
 $\cdot \eta : \{\sigma : \text{Sub } \Gamma \cdot\} \rightarrow \sigma = \epsilon$
 $\triangleright \beta_1 : \pi_1 (\sigma, t) = \sigma$
 $\triangleright \beta_2 : \pi_2 (\sigma, t) = t$
 $\triangleright \eta : (\pi_1 \sigma, \pi_2 \sigma) = \sigma$
 $\circ : (\sigma, t) \circ \delta = (\sigma \circ \delta, t[\delta])$
 $\text{U} : \text{Ty } \Gamma$

Displayed algebras

$\Gamma^D : \Gamma^A \rightarrow \text{Set}$
 $A^D : \Gamma^D \gamma \rightarrow A^A \gamma \rightarrow \text{Set}$
 $\sigma^D : \Gamma^D \gamma \rightarrow \Delta^D (\sigma^A \gamma)$
 $t^D : (\gamma^D : \Gamma^D \gamma) \rightarrow A^D \gamma^D (t^A \gamma)$
 $\cdot^D \text{tt} : \equiv \top$
 $(\Gamma \triangleright A)^D (\gamma, \alpha) : \equiv (\gamma^D : \Gamma^D \gamma) \times A^D \gamma^D \alpha$
 $(A[\sigma])^D \gamma^D \alpha : \equiv A^D (\sigma^D \gamma^D) \alpha$
 $\text{id}^D \gamma^D : \equiv \gamma^D$
 $(\sigma \circ \delta)^D \gamma^D : \equiv \sigma^D (\delta^D \gamma^D)$
 $\epsilon^D \gamma^D : \equiv \text{tt}$
 $(\sigma, t)^D \gamma^D : \equiv (\sigma^D \gamma^D, t^D \gamma^D)$
 $(\pi_1 \sigma)^D \gamma^D : \equiv \text{proj}_1 (\sigma^D \gamma^D)$
 $(\pi_2 \sigma)^D \gamma^D : \equiv \text{proj}_2 (\sigma^D \gamma^D)$
 $(t[\sigma])^D \gamma^D : \equiv t^D (\sigma^D \gamma^D)$
 $[\text{id}]^D : \equiv \text{refl}$
 $[\circ]^D : \equiv \text{refl}$
 $\text{ass}^D : \equiv \text{refl}$
 $\text{idl}^D : \equiv \text{refl}$
 $\text{idr}^D : \equiv \text{refl}$
 $\cdot \eta^D : \equiv \text{refl}$
 $\triangleright \beta_1^D : \equiv \text{refl}$
 $\triangleright \beta_2^D : \equiv \text{refl}$
 $\triangleright \eta^D : \equiv \text{refl}$
 $\circ^D : \equiv \text{refl}$
 $\text{U}^D \gamma^D T : \equiv T \rightarrow \text{Set}$

$\text{El } (a : \text{Tm } \Gamma \text{U}) : \text{Ty } \Gamma$
 $\text{U}[] : \text{U}[\sigma] = \text{U}$
 $\text{El}[] : (\text{El } a) [\sigma] = \text{El } (a[\sigma])$
 $\Pi (a : \text{Tm } \Gamma \text{U}) (B : \text{Ty } (\Gamma \triangleright \text{El } a)) : \text{Ty } \Gamma$
 $\text{app } (t : \text{Tm } \Gamma (\Pi a B)) : \text{Tm } (\Gamma \triangleright \text{El } a) B$
 $\Pi[] : (\Pi a B) [\sigma] = \Pi (a[\sigma]) (B[\sigma^\dagger])$
 $\text{app}[] : (\text{app } t) [\sigma^\dagger] = \text{app } (t[\sigma])$
 $\text{ld } (a : \text{Tm } \Gamma \text{U}) (t u : \text{Tm } \Gamma (\text{El } a)) : \text{Ty } \Gamma$
 $\text{reflect } (e : \text{Tm } \Gamma (\text{ld } a t u)) : t = u$
 $\text{ld}[] : (\text{ld } a t u) [\sigma] = \text{ld } (a[\sigma]) (t[\sigma]) (u[\sigma])$
 $\hat{\Pi} (T : \text{Set}) (B : T \rightarrow \text{Ty } \Gamma) : \text{Ty } \Gamma$
 $(t : \text{Tm } \Gamma (\hat{\Pi} T B)) \hat{\alpha} (\alpha : T) : \text{Tm } \Gamma (B \alpha)$
 $\hat{\Pi}[] : (\hat{\Pi} T B) [\sigma] = \hat{\Pi} T (\lambda \alpha. (B \alpha) [\sigma])$
 $\hat{\alpha}[] : (t \hat{\alpha} \alpha) [\sigma] = (t[\sigma]) \hat{\alpha} \alpha$

$(\text{El } a)^D \gamma^D \alpha : \equiv a^D \gamma^D \alpha$
 $\text{U}[]^A : \equiv \text{refl}$
 $\text{El}[]^A : \equiv \text{refl}$
 $(\Pi a B)^D \gamma^D f : \equiv (\alpha^D : a^D \gamma^D \alpha) \rightarrow B^D (\gamma^D, \alpha^D) (f \alpha)$
 $(\text{app } t)^D (\gamma^D, \alpha^D) : \equiv t^D \gamma^D \alpha^D$
 $\Pi[]^D : \equiv \text{refl}$
 $\text{app}[]^D : \equiv \text{refl}$
 $(\text{ld } a t u)^D \gamma^D e : \equiv \text{tr}_{(a^D \gamma^D)} e (t^D \gamma^D) = u^D \gamma^D$
 $(\text{reflect } e)^D : t^D \gamma^D \epsilon^D \stackrel{\gamma^D}{=} u^D \gamma^D$
 $\text{ld}[]^D : \equiv \text{refl}$
 $(\hat{\Pi} T B)^D \gamma^D f : \equiv (\alpha : T) \rightarrow (B \alpha)^D \gamma^D (f \alpha)$
 $(t \hat{\alpha} \alpha)^D \gamma^D : \equiv t^D \gamma^D \alpha$
 $\hat{\Pi}[]^D : \equiv \text{refl}$
 $\hat{\alpha}[]^D : \equiv \text{refl}$

$(_)^D : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

Displayed nat-algebras are “induction data”

$$\begin{aligned}\mathfrak{N}\text{-Alg} := & (N : \text{Set}) \\ & \times (z : N) \\ & \times (s : N \rightarrow N)\end{aligned}$$

$$\begin{aligned}\mathfrak{N}\text{-DAlg } (N, z, s) := & (N^{\text{D}} : N \rightarrow \text{Set}) \\ & \times (z^{\text{D}} : N^{\text{D}} \, z) \\ & \times (s^{\text{D}} : (n : N) \rightarrow N^{\text{D}} \, n \rightarrow N^{\text{D}}(s \, n))\end{aligned}$$

Question: What kind
of thing is the
output of induction?

Fact We can define the type of **sections** of a given displayed $\mathcal{G}$ -algebra, for all GATs $\mathcal{G}$.

Theorem Every displayed algebra over the initial $\mathcal{G}$ -algebra admits a section

- Principle of induction (e.g. $\mathbb{N}$)
- Syntax model is contextual
- Unary parametricity

$$\begin{aligned}
& _ - \text{Alg} : (\mathfrak{G} : \text{GAT}) \rightarrow \text{Set} \\
& \text{Hom} : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set} \\
& _ - \text{DAAlg} : (\mathfrak{G} : \text{GAT}) \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set} \\
& _ - \text{Sect} : (\mathfrak{G} : \text{GAT}) \rightarrow (\Gamma : \mathfrak{G}\text{-Alg}) \rightarrow \mathfrak{G}\text{-DAAlg } \Gamma \\
& \quad \rightarrow \text{Set}
\end{aligned}$$

From [KKA19, Appendix A]

Sections			
Γ^S	$: (\gamma : \Gamma^A) \rightarrow \Gamma^D \gamma \rightarrow \text{Set}$	$(\text{El } a)^S \gamma^S \alpha \alpha^D$	$\equiv a^S \gamma^S \alpha = \alpha^D$
A^S	$: \Gamma^S \gamma \gamma^D \rightarrow (\alpha : A^A \gamma) \rightarrow A^D \gamma^D \alpha \rightarrow \text{Set}$	$\cup []^S$	$\equiv \text{refl}$
σ^S	$: \Gamma^S \gamma \gamma^D \rightarrow \Delta^S (\sigma^A \gamma) (\sigma^D \gamma^D)$	$\text{El} []^S$	$\equiv \text{refl}$
t^S	$: (\gamma^S : \Gamma^S \gamma \gamma^D) \rightarrow A^S \gamma^S (t^A \gamma) (t^D \gamma^D)$	$(\Pi a B)^S \gamma^S f f^D$	$\equiv (\alpha : a^A \gamma) \rightarrow$ $B^S (\gamma^S, \text{refl}_{a^S \gamma^S \alpha}) (f \alpha) (f^D (a^S \gamma^S \alpha))$
tt^S	$\equiv \top$	$(\text{app } t)^S (\gamma^S, \alpha^S)$	$\equiv \bigvee_{x.z.B^S (\gamma^S, z)} (t^A \gamma \alpha) (t^D \gamma^D x) (t^S \gamma^S \alpha) \alpha^S$
$(\Gamma \triangleright A)^S (\gamma, \alpha) (\gamma^D, \alpha^D)$	$\equiv (\gamma^S : \Gamma^S \gamma \gamma^D) \times A^S \gamma^S \alpha \alpha^D$	$\Pi []^S$	$\equiv \text{refl}$
$(A[\sigma])^S \gamma^S \alpha \alpha^D$	$\equiv A^S (\sigma^S \gamma^S) \alpha \alpha^D$	$\text{app} []^S$	$\equiv \text{refl}$
$\text{id}^S \gamma^S$	$\equiv \gamma^S$	$(\text{Id } a t u)^S \gamma^S e e^D$	$\equiv \top$
$(\sigma \circ \delta)^S \gamma^S$	$\equiv \sigma^S (\delta^S \gamma^S)$	$(\text{reflect } e)^S$	$\equiv \text{UIP}$
$\epsilon^S \gamma^S$	$\equiv \text{tt}$	$\text{Id} []^S$	$\equiv \text{refl}$
$(\sigma, t)^S \gamma^S$	$\equiv (\sigma^S \gamma^S, t^S \gamma^S)$	$(\hat{\Pi} T B)^S \gamma^S f f^D$	$\equiv (\alpha : T) \rightarrow (B \alpha)^S \gamma^S (f \alpha) (f^D \alpha)$
$(\pi_1 \sigma)^S \gamma^S$	$\equiv \text{proj}_1 (\sigma^S \gamma^S)$	$(t \hat{\otimes} \alpha)^S \gamma^S$	$\equiv t^S \gamma^S \alpha$
$(\pi_2 \sigma)^S \gamma^S$	$\equiv \text{proj}_2 (\sigma^S \gamma^S)$	$\hat{\Pi} []^S$	$\equiv \text{refl}$
$(t[\sigma])^S \gamma^S$	$\equiv t^S (\sigma^S \gamma^S)$	$\hat{\otimes} []^S$	$\equiv \text{refl}$
$[\text{id}]^S$	$\equiv \text{refl}$		
$[\circ]^S$	$\equiv \text{refl}$		
ass^S	$\equiv \text{refl}$		

$$(_)^S : (\mathfrak{G} : \text{GAT}) \rightarrow (\Gamma : \mathfrak{G}\text{-Alg}) \rightarrow \mathfrak{G}\text{-DAlg} \rightarrow \text{Set}$$

$\triangleright \rho_1$	$\equiv \text{refl}$
$\triangleright \beta_2^S$	$\equiv \text{refl}$
$\triangleright \eta^S$	$\equiv \text{refl}$
$\circ^S$	$\equiv \text{refl}$
$\cup^S \gamma^S T T^D$	$\equiv (\alpha : T) \rightarrow T^D \alpha$

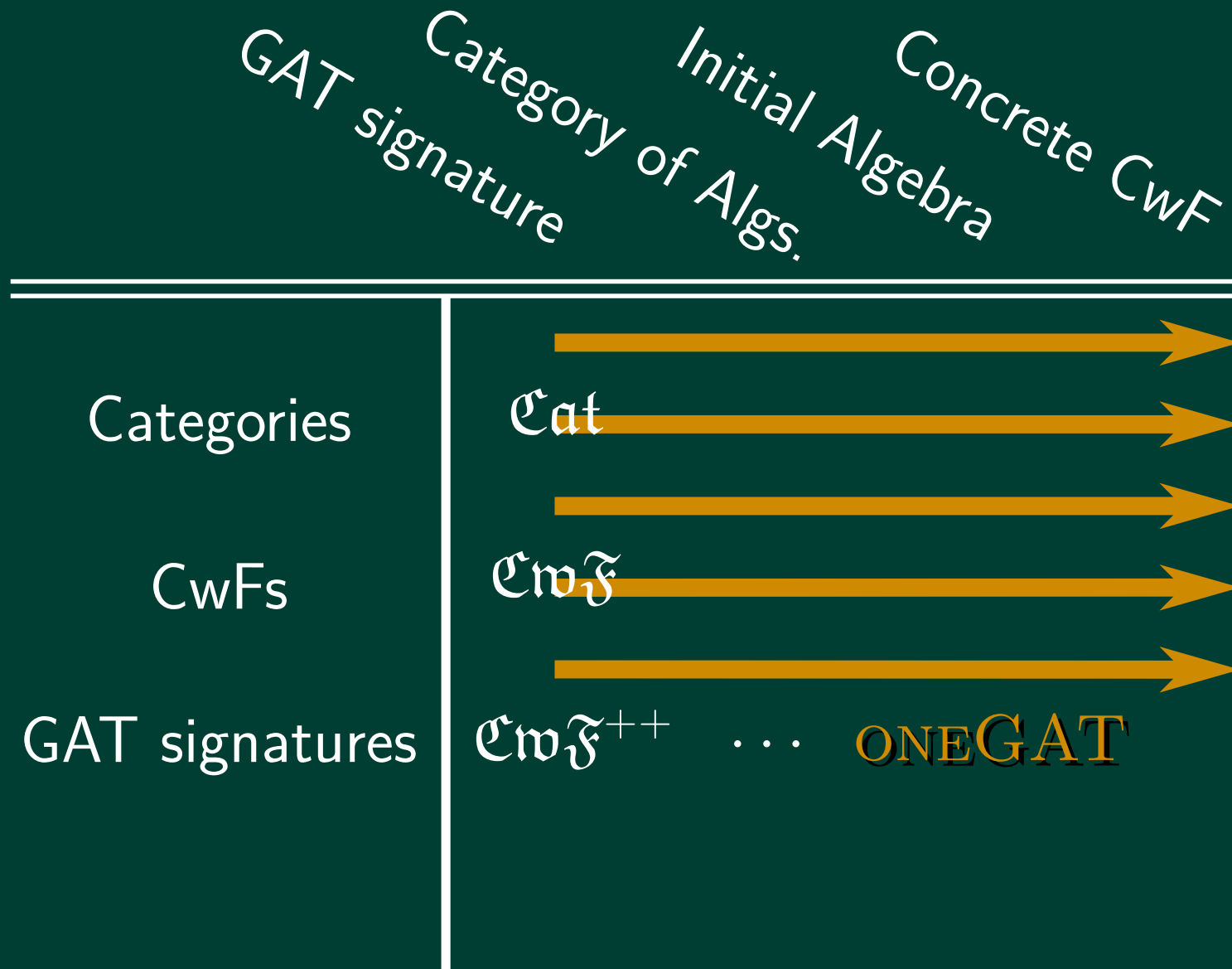
$\mathfrak{G}\text{-Alg} : \text{Set}$

$\text{Hom}_{\mathfrak{G}} : \mathfrak{G}\text{-Alg} \rightarrow \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\mathfrak{G}\text{-DAlg} : \mathfrak{G}\text{-Alg} \rightarrow \text{Set}$

$\mathfrak{G}\text{-Sect} : (\Gamma : \mathfrak{G}\text{-Alg}) \rightarrow \mathfrak{G}\text{-DAlg } \Gamma$
 $\rightarrow \text{Set}$

Observation: Every
GAT gives rise to a
CwF of algebras and
displayed algebras



Is the **setoid model**¹ the same thing as the concrete CwF of setoids?

Is the **groupoid model** the same thing as the concrete CwF of groupoids?

¹We mean the setoid model à la Altenkirch [Alt99], which uses **equivalence relations**, not the setoid model of Hofmann [Hof95a, Hof95b], which uses **partial equivalence relations**.

No

From [ABK⁺21, 7]

Displayed Setoid

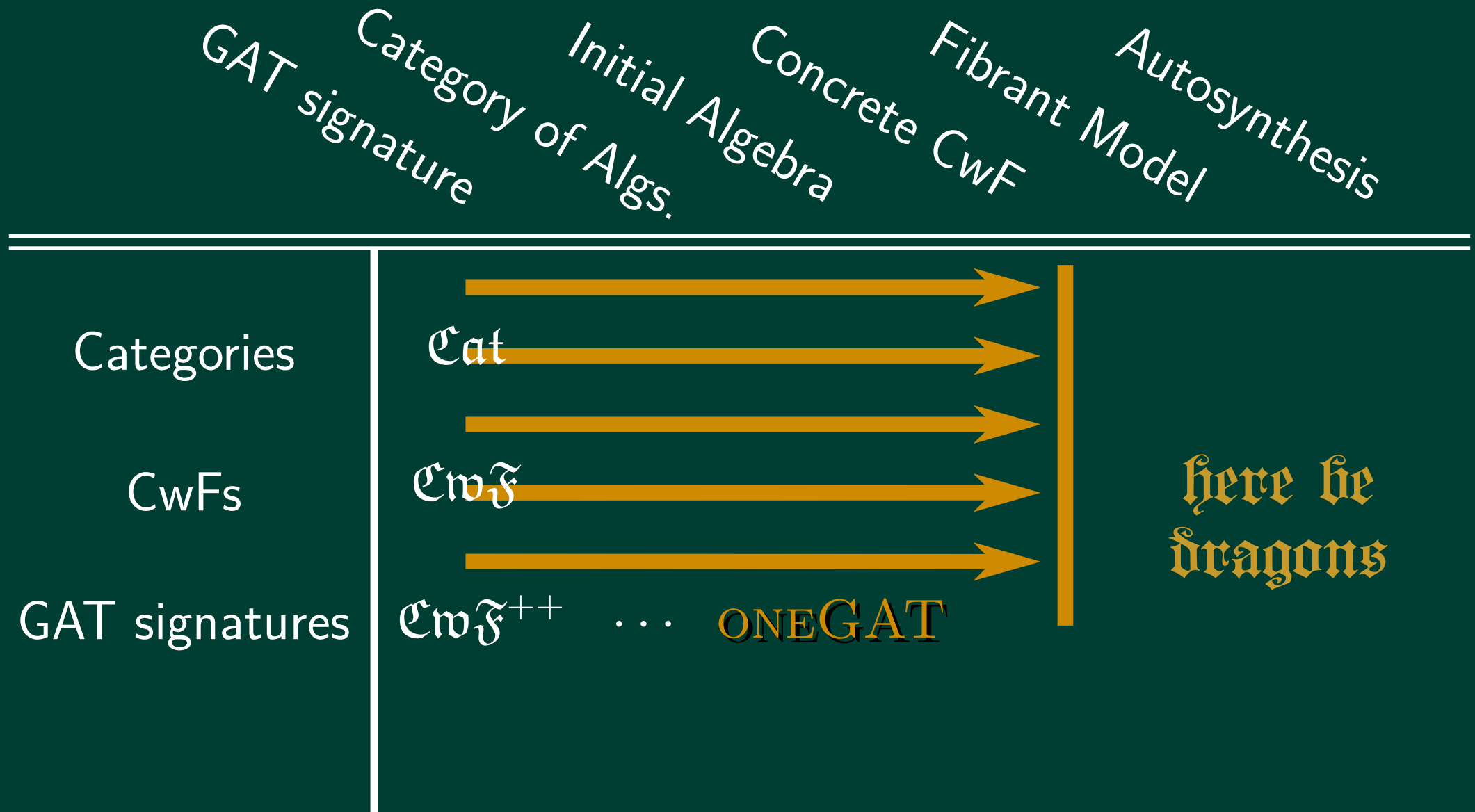
$A : \mathbf{Ty} \ \Gamma$

$$\begin{aligned} |A| &: |\Gamma| \rightarrow \mathbf{Type} \\ A^\sim &: \{\gamma_0 \ \gamma_1 : |\Gamma|\} \rightarrow \Gamma^\sim \ \gamma_0 \ \gamma_1 \rightarrow |A|_{\gamma_0} \rightarrow |A|_{\gamma_1} \rightarrow \mathbf{Prop} \\ \mathsf{refl}^* &: \{\gamma : |\Gamma|\} (a : |A|_\gamma) \rightarrow A^\sim \ (\mathsf{refl} \ \Gamma \ \gamma) \ a \ a \\ \mathsf{sym}^* &: \forall \{\gamma_0 \ \gamma_1 \ a_0 \ a_1\} \{p : \Gamma^\sim \ \gamma_0 \ \gamma_1\} \rightarrow A^\sim \ p \ a_0 \ a_1 \rightarrow A^\sim \ (\mathsf{sym} \ \Gamma \ p) \ a_1 \ a_0 \\ \mathsf{trans}^* &: A^\sim \ p_0 \ a_0 \ a_1 \rightarrow A^\sim \ p_1 \ a_1 \ a_2 \rightarrow A^\sim \ (\mathsf{trans} \ \Gamma \ p_0 \ p_1) \ a_0 \ a_2 \\ \mathsf{coe} &: \Gamma^\sim \ \gamma_0 \ \gamma_1 \rightarrow |A|_{\gamma_0} \rightarrow |A|_{\gamma_1} \\ \mathsf{coh} &: (p : \Gamma^\sim \ \gamma_0 \ \gamma_1) (a : |A|_{\gamma_0}) \rightarrow A^\sim \ p \ a \ (\mathsf{coe} \ A \ p \ a) \end{aligned}$$

Fibrancy

Idea: Carve out
sub-CwFs of
concrete CwFs whose
types are *fibrant*

- The groupoid model provides a synthetic theory of groupoids
- The setoid model provides a synthetic theory of setoids
- The preorder model provides a synthetic theory of preorders?
- The category model provides a synthetic theory of categories?
 - ▶ Yes: [Neu25]



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The End

(thank you!)