Polarity Problems

and the Semantics of Directed Type Theory

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0 Background

Identity Types

Standard Martin-Löf Type Theory includes **identity types**:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash t : A \quad \Gamma \vdash t' : A}{\Gamma \vdash \mathsf{Id}_A(t, t') \text{ type}}$$

$$\frac{\Gamma \text{ ctx} \quad \Gamma \vdash t : A}{\Gamma \vdash \mathsf{refl}_t : \mathsf{Id}(t, t)}$$

$$\frac{\begin{array}{c}\Gamma \vdash t : A \\ \Gamma, x : A, u : \mathsf{Id}(t, x) \vdash M(x, u) \text{ type} \\ \Gamma \vdash m : M(t, \mathsf{refl})\end{array}}{\Gamma, x : A, u : \mathsf{Id}(t, x) \vdash J\ m\ x\ u : M(x, u)}$$

Symmetry

$$\frac{\Gamma \vdash p \colon \mathsf{Id}(t, t')}{\Gamma \vdash p^{-1} := J\ \mathsf{refl}_t\ t'\ p \colon \mathsf{Id}(t', t)}$$

Transitivity

$$\frac{\Gamma \vdash p \colon \mathsf{Id}(t, t') \qquad \Gamma \vdash q \colon \mathsf{Id}(t', t'')}{\Gamma \vdash p \cdot q := J\ p\ t''\ q \colon \mathsf{Id}(t, t'')}$$

Idea The Id-types make each type into a **synthetic groupoid**

- "Synthetic": Groupoids (types) are fundamental objects; there's no proving *that something is a groupoid*

Question: Can we have a synthetic *category* theory?

Directed Type Theory includes **hom-types**:

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash t : A \qquad \Gamma \vdash t' : A}{\Gamma \vdash \mathrm{Hom}_{\mathbf{A}}(t, t') \text{ type}}$$

$$\frac{\Gamma \text{ ctx} \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathrm{refl}_t : \mathrm{Hom}(t, t)} \qquad \frac{\begin{array}{c}\Gamma \vdash t : A \\ \Gamma, x : A, u : \mathrm{Hom}(t, x) \vdash M(x, u) \text{ type} \\ \Gamma \vdash m : M(t, \mathrm{refl})\end{array}}{\Gamma, x : A, u : \mathrm{Hom}(t, x) \vdash J\ m\ x\ u : M(x, u)}$$

Symmetry

$$\frac{\Gamma \vdash p : \mathrm{Hom}(t, t')}{\Gamma \vdash J\ \mathrm{refl}_t\ t'\ p : \mathrm{Hom}(t', t)}$$

✗

Transitivity

$$\frac{\Gamma \vdash p : \mathrm{Hom}(t, t') \qquad \Gamma \vdash q : \mathrm{Hom}(t', t'')}{\Gamma \vdash J\ p\ t''\ q : \mathrm{Hom}(t, t'')}$$

✓

Key Idea: Need *semantics* to prove that symmetry is unprovable

Hofmann and Streicher prove [HS95] that the *uniqueness of identity proofs (UIP)* is *independent from the rules of Martin-Löf Type Theory*.

1. Interpret the syntax of Martin-Löf (contexts, substitutions, types, terms, etc.) as mathematical structures (groupoids, families of groupoids, etc.)
2. Prove that the rules of MLTT (e.g. the J-rule) are valid in this interpretation
3. Prove that UIP is *not* valid in this interpretation
4. Deduce that the rules of MLTT *cannot* prove MLTT

Idea: Replace groupoids with categories

The approach

1. Define the **category model**
2. Figure out what syntax of *directed type theory* it interprets
3. Prove that symmetry is independent of this directed type theory

1 Shallow and Deep Polarity

Basic Structure of Type Theory

- Contexts are categories
- Substitutions are functors
- Types in context Γ are Γ-indexed families of categories (Grothendieck opfibrations)

$$\mathsf{Con} = \mathsf{Cat}$$

$$\mathsf{Sub} : \mathsf{Con} \to \mathsf{Con} \to \mathsf{Set}$$
$$\mathsf{Sub}\ \Delta\ \Gamma := \Delta \to \Gamma$$

$$\mathsf{Ty} : \mathsf{Con} \to \mathsf{Set}$$
$$\mathsf{Ty}\ \Gamma := \Gamma \to \mathsf{Cat}$$

$$\_[\_] : \mathsf{Ty}\ \Gamma \to \mathsf{Sub}\ \Delta\ \Gamma \to \mathsf{Ty}\ \Delta$$
$$A\,[\,\sigma\,] := A \circ \sigma$$

- Terms are sections
- Context extension is the Grothendieck construction

record Tm (Γ : Con) (A : Ty Γ) : Set
  where
    obj : $(\gamma : |\Gamma|) \to |A\ \gamma|$
    mor : $(\gamma_{01} : \Gamma[\ \gamma_0\ ,\ \gamma_1\ ]) \to$
    $(A\ \gamma_1)[\ A\ \gamma_{01}\ (\text{obj}\ \gamma_0)\ ,\ \text{obj}\ \gamma_1\ ]$
    _ : mor id = id
    _ : mor $(\gamma_{12} \circ \gamma_{01}) =$
    $(\text{mor}\ \gamma_{12}) \circ A\ \gamma_{12}\ (\text{mor}\ \gamma_{01})$

_▷_ : (Γ : Con) → Ty Γ → Con
  $|\ \Gamma \triangleright A\ | := (\gamma : |\Gamma|) \times |A\ \gamma|$
  $(\Gamma \triangleright A)[\ (\gamma_0{,}a_0)\ ,\ (\gamma_1{,}a_1)\ ] :=$
    $(\gamma_{01} : \Gamma[\gamma_0{,}\gamma_1]) \times$
    $(A\ \gamma_1)[\ A\ \gamma_{01}\ a_0\ ,\ a_1\ ]$

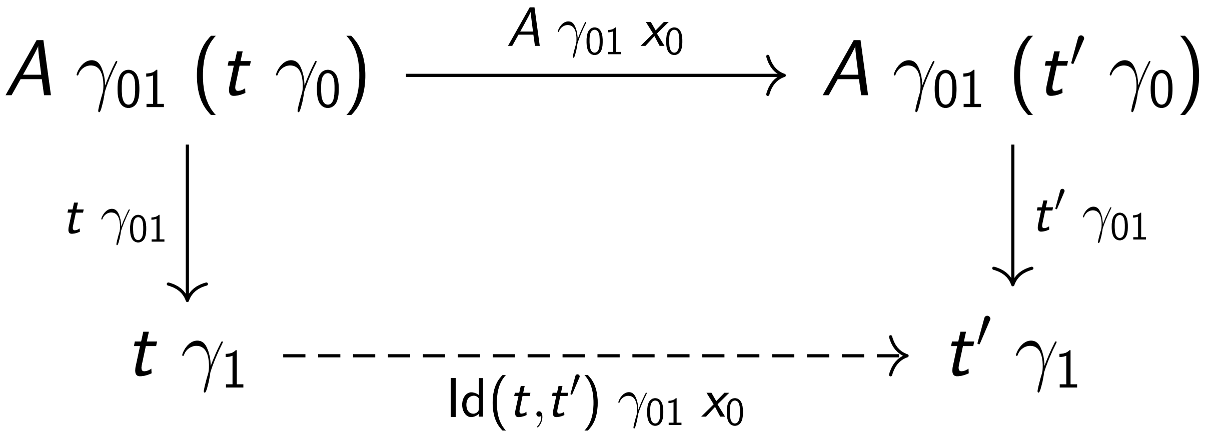
Polarity Problem:

A place where the groupoid model actually relies on the semantics being *groupoids*

Problem No. 1:
Hom-formation

Given $t\colon \mathrm{Tm}(\Gamma, A)$ and $t'\colon \mathrm{Tm}(\Gamma, A)$

$$
\begin{aligned}
\mathrm{Id}(t, t') &: \Gamma \to \mathbf{Grpd} \\
\mathrm{Id}(t, t')\ \gamma &= (A\ \gamma)\ [t\ \gamma, t'\ \gamma] \\
\mathrm{Id}(t, t')\ \gamma_{01} &: (A\ \gamma_0)\ [t\ \gamma_0, t'\ \gamma_0] \to (A\ \gamma_1)\ [t\ \gamma_1, t'\ \gamma_1] \\
\mathrm{Id}(t, t')\ \gamma_{01}\ x_0 &= (t'\ \gamma_{01}) \circ A\ \gamma_{01}\ x_0 \circ (t\ \gamma_{01})^{-1}
\end{aligned}
$$



We annotate **variances/polarities** by adding operations to MLTT

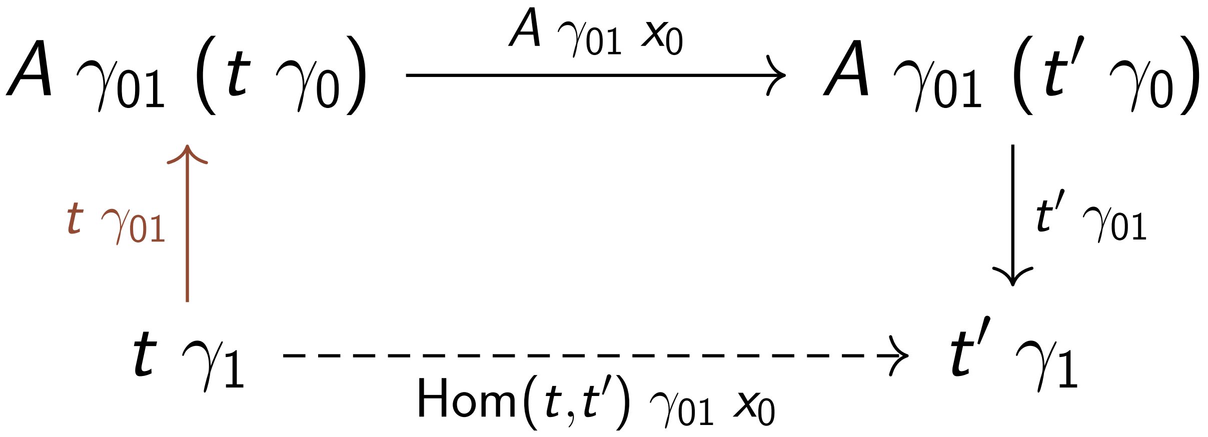
- **Type negation**

$$\frac{\Gamma \vdash A \text{ type}}{\Gamma \vdash A^{-} \text{ type}} \qquad \frac{}{(A^{-})^{-} = A}$$

$$\Gamma \xrightarrow{A} \mathbf{Cat} \xrightarrow{(\_)^{\mathrm{op}}} \mathbf{Cat}$$

Given $t : \mathrm{Tm}(\Gamma, A^{-})$ and $t' : \mathrm{Tm}(\Gamma, A)$

$$
\begin{aligned}
&\mathrm{Hom}(t, t') &&: \Gamma \to \mathbf{Cat} \\
&\mathrm{Hom}(t, t')\ \gamma &&= (A\ \gamma)\ [t\ \gamma, t'\ \gamma] \\
&\mathrm{Hom}(t, t')\ \gamma_{01} &&: (A\ \gamma_0)\ [t\ \gamma_0, t'\ \gamma_0] \to (A\ \gamma_1)\ [t\ \gamma_1, t'\ \gamma_1] \\
&\mathrm{Hom}(t, t')\ \gamma_{01}\ x_0 &&= (t'\ \gamma_{01}) \circ A\ \gamma_{01}\ x_0 \circ (t\ \gamma_{01})
\end{aligned}
$$



“Shallow polarity”

Problem No. 2:
Pi-Types

From [HS95, Section 4.6]:

If $\gamma \in \Gamma$ let $B_\gamma \in Ty(A(\gamma))$ be the family of groupoids over the groupoid $A(\gamma)$ given by

$$B_\gamma(a) = B(\gamma, a)$$
$$B_\gamma(p)(\_) = (id_\gamma, p) \cdot \_$$

Notice that $B_\gamma = B\{\hat{\gamma}\}$ where $\hat{\gamma} : A(\gamma) \to \Gamma.A$ is the functor sending a to (γ, a) and $p : a \to a'$

Morphisms of Γ, inverted

Now we put

$$\Pi_{\mathrm{LF}}(A, B)(\gamma) = Tm(B_\gamma) \quad \text{considered as a groupoid}$$

If $p : \gamma \to \gamma'$ and $M \in Tm(B_\gamma)$ then $(p \cdot M) \in Tm(B'_\gamma)$ is given by

$$(p \cdot M)(a \in A(\gamma')) = (p, id) \cdot M(p^{-1} \cdot a)$$
$$(p \cdot M)(q : a \to a') = (p, id) \cdot M(p^{-1} \cdot q)$$

“Deep polarity”

We annotate **variances/polarities** by adding operations to MLTT

- **Context negation**

$$\frac{\Gamma \text{ ctx}}{\Gamma^{-} \text{ ctx}} \qquad \frac{}{(\Gamma^{-})^{-} = \Gamma}$$

$\Gamma^{-} := \Gamma^{\mathrm{op}}$

- **Negative context extension**

$$\frac{\Gamma^{-} \vdash A \text{ type}}{\Gamma ,_{-} x : A \text{ ctx}} \qquad \frac{\Gamma^{-} \vdash A \text{ type}}{(\Gamma ,_{-} x : A)^{-} \vdash x : A^{-}}$$

Polarizing Pi types

$$\frac{\Gamma^{-} \vdash A \text{ type} \qquad \Gamma ,_{-}\ x : A \vdash B \text{ type}}{\Gamma \vdash \Pi(A, B) \text{ type}}$$

$$\frac{\Gamma ,_{-}\ x : A \vdash e : B}{\Gamma \vdash \lambda x.e : \Pi(A, B)} \qquad \frac{\Gamma \vdash f : \Pi(A, B) \qquad \Gamma^{-} \vdash t : A^{-}}{\Gamma \vdash f(t) : B(t)}$$

Problem: We can’t do anything with these

2 Neutrality

- No identity function
  - $A \to A$ isn't a well-formed type (domain has to be in Γ^-, codomain in $\Gamma ,_- x\colon A$)
  - $\lambda x.x$ isn't a well-formed term (x is a term in $(\Gamma ,_- x : A)^-$, not $\Gamma ,_- x\colon A$)
- Forget about composition...
- Can't introduce or eliminate hom-types

Hom rules

$$\frac{\Gamma \vdash A \text{ type} \qquad \Gamma \vdash t : A^{-} \qquad \Gamma \vdash t' : A}{\Gamma \vdash \mathrm{Hom}_A(t, t') \text{ type}}$$

$$\frac{\Gamma \text{ ctx} \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathrm{refl}_t : \mathrm{Hom}(\mathbf{t}, \mathbf{t})} \qquad \frac{\begin{array}{c}\Gamma \text{ ctx} \qquad\qquad \Gamma \vdash t : A \\ \Gamma, \mathbf{x} : \mathbf{A}, u : \mathrm{Hom}(t, x) \vdash M(x, u) \text{ type} \\ \Gamma \vdash m : M(\mathbf{t}, \mathrm{refl})\end{array}}{\Gamma, x : A, u : \mathrm{Hom}(\mathbf{t}, x) \vdash J\ m\ x\ u : M(x, u)}$$

"Overly-strict polarity"

We instead work in a **neutral context**, i.e. a groupoid (but types are still valued in categories)

- Can substitute with the isomorphism $\mathrm{e} \colon \Gamma^{\mathrm{op}} \cong \Gamma$ (so types like $A \to A$ can be well-formed as $A[\mathrm{e}] \to A$)
- Can substitute between $\Gamma ,_{-}\ x \colon A[\mathrm{e}]$ and $\Gamma, x \colon A$ (so $\lambda x.x$ is well-formed)
- Get coercion operator

$$\frac{\Gamma\ \mathrm{Nctx} \qquad \Gamma \vdash t \colon A^{-}}{\Gamma \vdash -t \colon A} \qquad\qquad - -t = t$$

$- : \{\Gamma : \mathrm{NeutCon}\}\{A : \mathrm{Ty}\ \Gamma\} \to \mathrm{Tm}(\Gamma,A) \to \mathrm{Tm}(\Gamma,A^{-})$
$-t'\ \gamma = t'\ \gamma$
$-t'\ \gamma_{01} = A\ \gamma_{01}\ (t'({\gamma_{01}}^{-1}))$

Hom rules

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma \vdash t : A^{-} \quad \Gamma \vdash t' : A}{\Gamma \vdash \mathrm{Hom}_A(t, t') \text{ type}}$$

$$\frac{\Gamma \ \mathsf{Nctx} \quad \Gamma \vdash t : A^{-}}{\Gamma \vdash \mathrm{refl}_t : \mathrm{Hom}(t, -t)}$$

$$\frac{\begin{array}{c}\Gamma \ \mathsf{Nctx} \qquad \Gamma \vdash t : A^{-} \\ \Gamma, x : A, u : \mathrm{Hom}(t, x) \vdash M(x, u) \text{ type} \\ \Gamma \vdash m : M(-t, \mathrm{refl})\end{array}}{\Gamma, x : A, u : \mathrm{Hom}(t, x) \vdash J\ m\ x\ u : M(x, u)}$$

$$\begin{aligned}
&J^+ : (t : \mathrm{Tm}(\Gamma, A^-)) \\
&\quad \to (M : \mathrm{Ty}\ (\Gamma \rhd^+ A \rhd^+ \mathrm{Hom}(t[\ p\ ], v_0))) \\
&\quad \to \mathrm{Tm}(\Gamma, M[\ id\ ,_+ -t\ ,_+ \mathrm{refl}_t\ ]) \\
&\quad \to \mathrm{Tm}(\Gamma \rhd^+ A \rhd^+ \mathrm{Hom}(t[\ p\ ], v_0), M)
\end{aligned}$$

$$\begin{aligned}
&(J^+{}_{t,M}\ m) : (\gamma : |\Gamma|) \to (a : |A\ \gamma|) \to (x : (A\ \gamma)\ [t\ \gamma\ , a]) \to |\ M(\gamma, a, x)\ | \\
&(J^+{}_{t,M}\ m)\ \gamma\ a\ x = M\ (id_\gamma, x)\ (m\ \gamma) \quad -\!-\ x = x \circ A\ id_\gamma\ id_{t\gamma} \circ t\ id_\gamma
\end{aligned}$$

$$\begin{aligned}
&(J^+{}_{t,M}\ m) : (\gamma_{01} : \Gamma\ [\gamma_0,\gamma_1]) \\
&\qquad \to (a_{01} : A\gamma_1\ [A\ \gamma_{01}\ a_0,\ a_1]) \\
&\qquad \to M(\gamma_1,\ a_1,\ a_{01} \circ A\ \gamma_{01}\ x_0 \circ t(\gamma_{01})\ )[\\
&\qquad\qquad M(\gamma_{01},a_{01})\ ((J^+{}_{t,M}\ m)\ \gamma_0\ a_0\ x_0), \\
&\qquad\qquad ((J^+{}_{t,M}\ m)\ \gamma_1\ a_1\ (a_{01} \circ A\ \gamma_{01}\ x_0 \circ t(\gamma_{01}))) \\
&\qquad\ ] \\
&(J^+{}_{t,M}\ m)\ \gamma_{01}\ a_{01} = M\ (id_{\gamma_1},\ a_{01} \circ A\ \gamma_{01}\ x_0 \circ t(\gamma_{01}))\ (m\ \gamma_{01})
\end{aligned}$$

3 Directed Type Theory

Some general notes

- Can prove that hom-types aren't provably symmetric (symmetry is independent of this directed type theory)
- Have notion of *neutral types* ($\mathsf{NeutTy}\ \Gamma := \Gamma \to \mathbf{Grpd}$). Neutral types have provably symmetric hom-types (by J^+), so we write their hom-types as Id.
- In the category model, every hom-type is neutral and satisfies UIP, i.e. is a *set* (call this (1,1)-directedTT)

$$\frac{\Gamma\ \mathsf{Nctx} \qquad \Gamma \vdash t : A^{-} \qquad \Gamma, x : A, u : \mathrm{Hom}(t, x) \vdash M(x, u)\ \mathsf{type} \qquad \Gamma \vdash m : M(-t, \mathsf{refl})}{\Gamma, x : A, u : \mathrm{Hom}(t, x) \vdash J\ m\ x\ u : M(x, u)}$$

$$\frac{\Gamma\ \mathsf{Nctx} \qquad \Gamma \vdash t : A^{-} \qquad \Gamma \vdash t' : A \qquad \Gamma \vdash f : \mathrm{Hom}(t, t') \qquad \Gamma, x : A, u : \mathrm{Hom}(-\mathbf{t}', x) \vdash \mathrm{Hom}(\mathbf{t}, \mathbf{x})\ \mathsf{type} \qquad \Gamma \vdash \mathbf{f} : \mathrm{Hom}(\mathbf{t}, \mathbf{t}')}{\Gamma, x : A, u : \mathrm{Hom}(-\mathbf{t}', x) \vdash J\ \mathbf{f}\ x\ u : \mathrm{Hom}(\mathbf{t}, \mathbf{x})}$$

$$f \cdot g := \mathrm{J}^{+}\ f\ t''\ g$$

Martin Hofmann and Thomas Streicher.
The groupoid interpretation of type theory.
Twenty-five years of constructive type theory (Venice, 1995), 36:83–111, 1995.

Jacob Neumann and Thorsten Altenkirch.
Synthetic 1-categories in directed type theory.
arXiv preprint arXiv:2410.19520, 2024.

Jacob Neumann.
A Generalized Algebraic Theory of Directed Equality.
PhD thesis, University of Nottingham, 2025.

Conclusion

- Semantics of type theory in **Cat**
- Adopt a system of **polarity**, annotating variances
- Too much polarity, need to weaken somewhat with **neutrality**
- Directed J-rule, which *can* prove transitivity/composition, but not symmetry/inverses
- Synthetic category theory, directed homotopy theory, concurrency and rewriting

Thank you!