Synthetic-Inductive Category Theory

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This talk corresponds to [Neu25, Chapter 4], draft available at

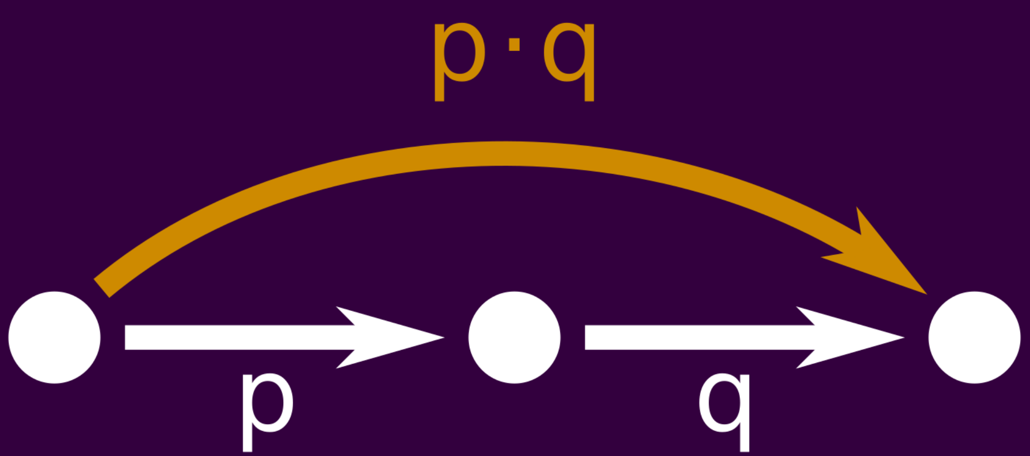
jacobneu.com/PhD

Martin Löf Type Theory: Synthetic groupoid theory

Synthetic groupoid structure of identity types

$$\frac{t : A}{\mathsf{refl}_t : \mathsf{Id}(t, t)}$$

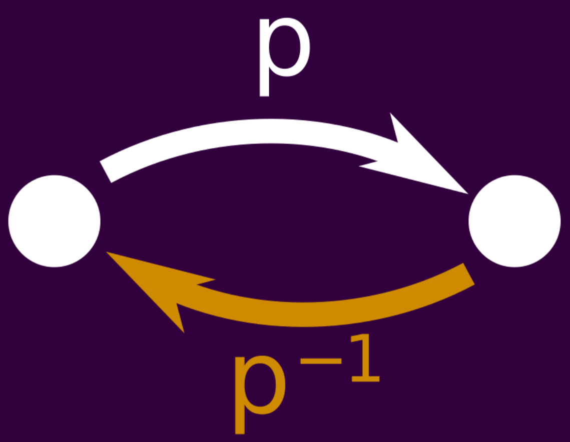
Synthetic groupoid structure of identity types



$$\frac{p : \mathsf{Id}(t, t') \qquad q : \mathsf{Id}(t', t'')}{p \cdot q : \mathsf{Id}(t, t'')}$$

$$p \cdot q := \mathsf{J}_{\mathsf{Id}(t,\_)}\, p\, (t'', q)$$

Synthetic groupoid structure of identity types



$$\frac{p\colon \mathsf{Id}(t, t')}{p^{-1}\colon \mathsf{Id}(t', t)}$$

$$p^{-1} := \mathsf{J}_{\mathsf{Id}(\_, t)}\ \mathsf{refl}_t\ (t', p)$$

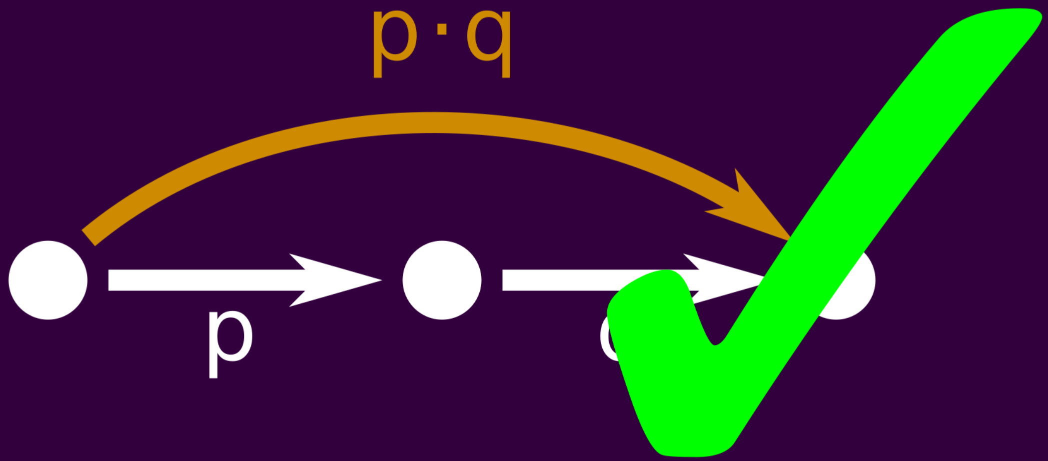
Directed TT:

Synthetic *category* theory

Synthetic category structure of hom-types

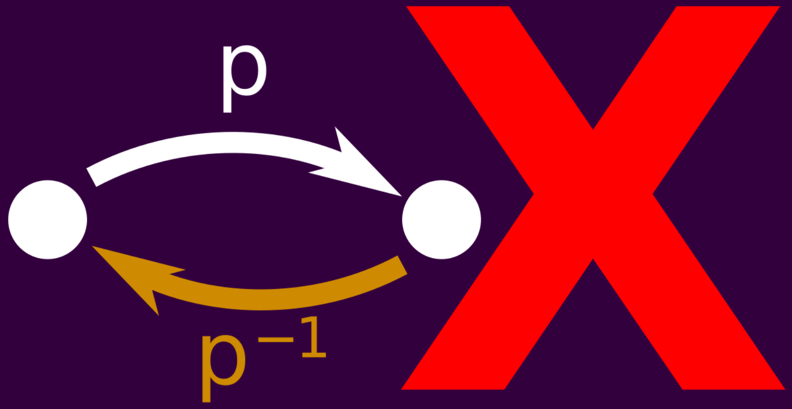
$$\frac{t : A^{-}}{\mathsf{refl}_t : \mathrm{Hom}(t, -t)}$$

Synthetic category structure of hom-types



$$\frac{p : \mathrm{Hom}(t, t') \qquad q : \mathrm{Hom}(-t', t'')}{p \cdot q : \mathrm{Hom}(t, t'')}$$

$$p \cdot q := \mathrm{J}_{\mathrm{Hom}(t,\_)}\; p\; (t'', q)$$



$$\frac{p : \mathrm{Hom}(t, t')}{p^{-1} : \mathrm{Hom}(-t', -t)}$$

$$p^{-1} := \mathrm{J}_{\mathrm{Hom}(\_, t)}\ \mathrm{refl}_t\ (t', p)$$

Problem of Directed TT: **Make symmetry unprovable**

Synthetic categories that aren't (necessarily) synthetic groupoids

We have **polarity annotations** on our types to mark co- or contra-variance

$$\frac{\Delta \vdash A \ \textbf{type}}{\Delta \vdash A^- \ \textbf{type}} \qquad \frac{}{\Delta \vdash (A^-)^- = A}$$

Types are categories, A^- is the opposite category of A

The polarity annotations allow us to properly state the variance of hom-sets [Nor19]:

$$\frac{\Delta \vdash t : A^- \qquad \Delta \vdash t' : A}{\Delta \vdash \mathrm{Hom}(t, t') \ \textbf{type}}$$

Closed term coercion

For **closed** terms, we can coerce between A and A^{-} [NA25, Neu25]:

$$\frac{t : A^{-}}{-t : A} \qquad \frac{}{- - t = t}$$

A category and its opposite have the same objects

Key Point Cannot (in general) negate open terms

$$x : A^{-} \vdash -x : A$$

For **closed** terms, we can coerce between A and A^- [NA25, Neu25]:

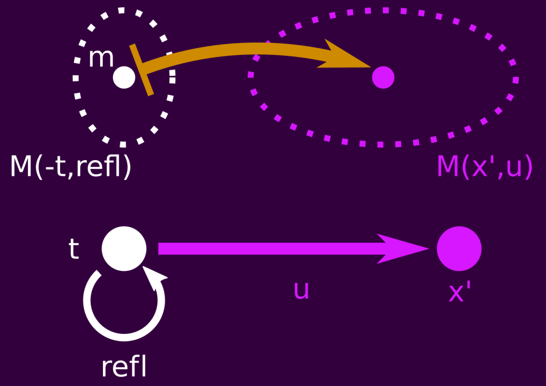
$$\frac{t : A^-}{-t : A} \qquad \frac{}{- - t = t}$$

The coercions make it possible to introduce refl:

$$\frac{t : A^-}{\mathrm{refl}_t : \mathrm{Hom}(t, -t)}$$

Synthetic category structure of hom-types

$$\frac{t : A^{-}}{\mathrm{refl}_t : \mathrm{Hom}(t, -t)}$$



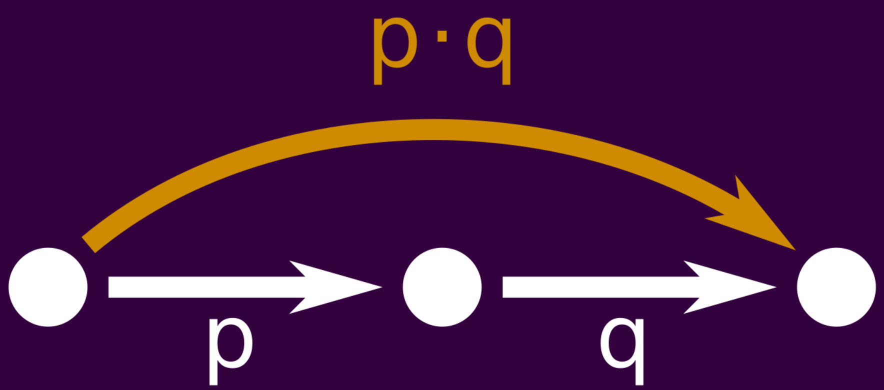
Coslice Path Induction

$$\frac{t : A^{-} \qquad x' : A, u : \mathrm{Hom}(t, x') \vdash M(x', u)\ \textbf{type} \qquad m : M(-t, \mathrm{refl}_t)}{x' : A, u : \mathrm{Hom}(t, x') \vdash \mathrm{J}^{+}_{M}\ m\ (x', u) : M(x', u)}$$

$$\mathrm{J}^{+}_{M}\ m\ (-t, \mathrm{refl}_t) = m$$

refl_t is the "universal coslice" under t

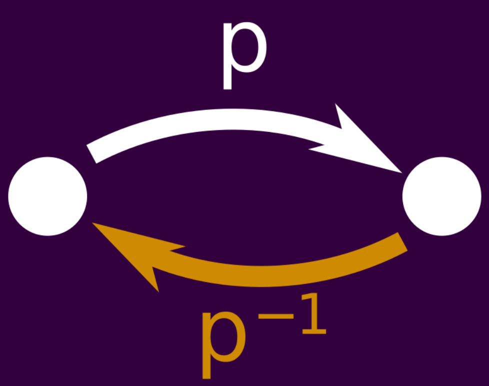
This solves the fundamental problem of directed TT



$$\frac{p : \mathrm{Hom}(t, t') \qquad q : \mathrm{Hom}(-t', t'')}{p \cdot q : \mathrm{Hom}(t, t'')}$$

$$x'' : A, u : \mathrm{Hom}(-t', x'') \vdash$$
$$\mathrm{J}^{+}_{\mathrm{Hom}(t, x'')}\, p\, (x'', u) : \mathrm{Hom}(t, x'')$$

Synthetic category structure of hom-types



$$\frac{p : \mathrm{Hom}(t, t')}{p^{-1} : \mathrm{Hom}(-t', -t)}$$

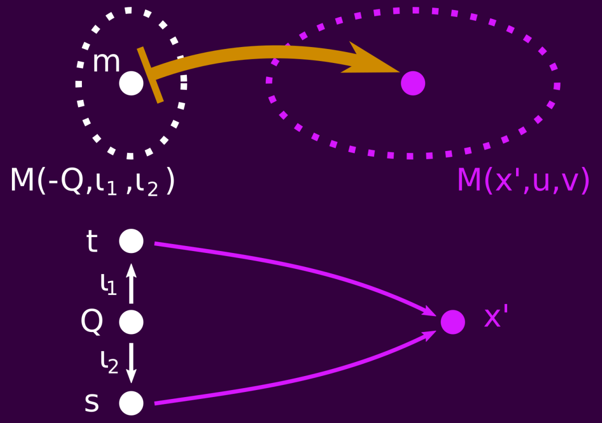
$$x' : A, u : \mathrm{Hom}(t, x') \vdash$$
$$\mathrm{J}^{+}\ \mathrm{refl}_t\ (x', u) : \mathrm{Hom}(-x', -t)$$

Semantic Proof:

Symmetry *can't* be proved in general

Category theory is concerned with
universal mapping properties

Type-theoretic constructs are introduced with
principles of induction



A **coproduct** of $s, t : A^{-}$ consists of terms

- $Q : A^{-}$
- $\iota_1 : \mathrm{Hom}(s, -Q)$ and
  $\iota_2 : \mathrm{Hom}(t, -Q)$

such that

$$\frac{x' : A, u : \mathrm{Hom}(s, x'), v : \mathrm{Hom}(t, x') \vdash M \textbf{ type} \qquad m : M(-Q, \iota_1, \iota_2)}{x', u, v \vdash \mathrm{elim}_M\ m\ (x', u, v) : M(x', u, v)}$$

$$\mathrm{elim}_M\ m\ (-Q, \iota_1, \iota_2) = m$$

A term $F\colon A \to B$ is a functor from A to B

- Define the identity functor on A, $I_A := \lambda x.x$.
- Given $F\colon A \to B$ and $f\colon \mathrm{Hom}_A(t, t')$, define map $F\ f\colon \mathrm{Hom}(-F(-t), F(t'))$:

$$x'\colon A, u\colon \mathrm{Hom}_A(t, x') \vdash \mathrm{J}^+\ \mathrm{refl}_{-F(-t)}\colon \mathrm{Hom}(-F(-t), F(x'))$$

A term $\alpha\colon \mathrm{Hom}_{A\to B}(F, G)$ is a natural transformation

By Coslice Path Induction, $\mathrm{refl}@t' := \mathrm{refl}_{-((-F)\ t')}$ yields

$$\frac{F\colon (A \to B)^- \quad G\colon A \to B \quad \alpha\colon \mathrm{Hom}(F, G) \quad t'\colon A}{\alpha@t'\colon \mathrm{Hom}(-((-F)\ t'), G(t'))}$$

A **left adjoint** of $U\colon B \to A$ consists of

- $F\colon A \to B$
- $\eta\colon \mathrm{Hom}(-I_A, U \circ F)$

such that

$$\frac{\begin{array}{c} t\colon A^{-} \\ z'\colon A, u\colon \mathrm{Hom}(t, U(z')) \vdash M(z', u)\ \textbf{type} \\ m\colon M(F(-t), \eta\,@\,(-t)) \end{array}}{z'\colon A, u\colon \mathrm{Hom}(t, U(z')) \vdash \mathrm{elim}_M\ m\ (z', u)\colon M(z', u)}$$

$$\mathrm{elim}_M\ m\ (F(-t), \eta\,@\,(-t)) = m$$

Status

Apprehended

- Initial and Terminal Objects
- (Co)Products
- Pullbacks and Pushouts
- Left and Right Adjoints
- Applying a natural transformation
- Definition of Yoneda embedding
  - Exponentials
  - *All* limits of shape I

Still at large

- Lambda-abstraction rule for natural transformations
  - Proof by directed path induction that natural transformations are natural
  - Internal proof of Yoneda Lemma
  - (Co)limits in presheaf categories
- Monomorphisms/Epimorphisms (coinductive characterization?)

[NA25] Jacob Neumann and Thorsten Altenkirch.
Synthetic 1-categories in directed type theory.
In Rasmus Ejlers Møgelberg and Benno van den Berg, editors, *30th International Conference on Types for Proofs and Programs (TYPES 2024)*, Leibniz International Proceedings in Informatics (LIPIcs), 2025.

[Neu25] Jacob Neumann.
A Generalized Algebraic Theory of Directed Equality.
PhD thesis, University of Nottingham, 2025.

[Nor19] Paige Randall North.
Towards a directed homotopy type theory.
Electronic Notes in Theoretical Computer Science, 347:223–239, 2019.

- Track polarities
- Limit coercions to closed terms
- Fail to prove symmetry
- Phrase universal mapping properties as principles of induction

Thank you!