# Dual-Context Directed Type Theory

***for Synthetic (Co)-Inductive Category Theory***


Jacob Neumann
Reykjavik University



25 October 2025
CT Octoberfest 2025

*A Judgmental Construction of Directed Type Theory*

[Neu25b]

arxiv.org/abs/2510.17494

*A Generalized Algebraic Theory of Directed Equality*

[Neu25a]

jacobneu.phd

# 0 Directed TT for Synthetic CT

# Analytic-Synthetic Distinction: Geometry

**Analytic**

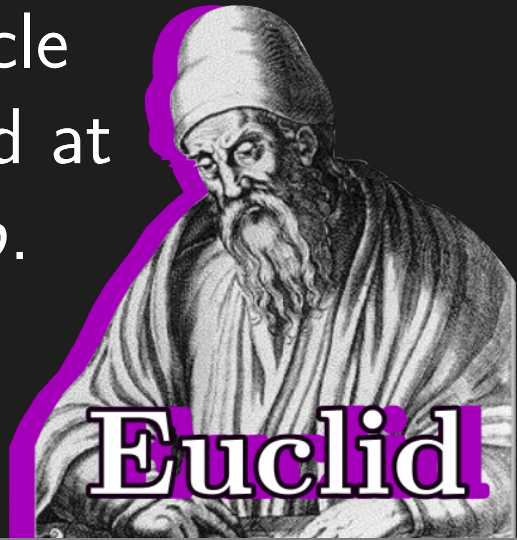
Given a point $c \in \mathbb{R} \times \mathbb{R}$ and $r > 0$, define

$$\{z \in \mathbb{R} \times \mathbb{R} \mid d(c, z) = r\}$$

**Synthetic**

There are things called points, lines, and circles.

Given any two (distinct) points $a$ and $b$, a circle may be drawn centered at $a$ which intersects $b$.

# Idea: Synthetic Category Theory

# Types are categories

| | |
|---|---|
| $A$ **type** | $A$ is a category |
| $s \colon A$ | $s$ is an object of $A$ |
| $$\frac{s \colon A \quad t \colon A}{\mathrm{Hom}(s,t)\ \textbf{type}}$$ | there is a set of morphisms between every two objects $s$ and $t$ of $A$ |

$$\frac{F \colon A \to B \quad f \colon \mathrm{Hom}_A(s,t)}{\mathrm{map}_F\ f \colon \mathrm{Hom}_B(F(s),F(t))}$$

$$\frac{F\ G \colon A \to B \quad \alpha \colon \mathrm{Hom}_{A \to B}(F,G)}{\alpha @ s \colon \mathrm{Hom}_B(F(s),G(s))}$$

In formal languages like type theory, we reason about *context dependency*

$$x_1 : A_1, \ldots, x_n : A_n \vdash b(x_1, \ldots, x_n) : B(x_1, \ldots, x_n)$$

If the types $A_1, \ldots, A_n, B$ have the structure of *synthetic categories*, this dependence is (dependent) **functoriality**

$$x : A^{-}, z : A \vdash \mathrm{Hom}(x, z)\ \textbf{type}$$

Hom depends *contravariantly* on $x$.

$$\frac{\Gamma \vdash A\ \textbf{type}}{\Gamma \vdash A^{-}\ \textbf{type}} \qquad \frac{}{\Gamma \vdash (A^{-})^{-} \equiv A}$$

# Problem: Identity morphisms

Logicians have developed theories with **multi-zoned contexts**, where the variables are split into different **zones** with different kinds of dependency

$$y_1 :: A_1, \ldots, y_m :: A_m \mid x_1 : B_1, \ldots, x_n : B_n \vdash c : C$$

In particular, [Shu18] is dealing with a type theory whose types are *synthetic topological spaces*. In the judgment above, the construction $c : C$ has to be continuous with respect to the "cohesive" variables $x_i : B_i$, but *not* with respect to the "crisp" variables $y_i :: A_i$.

$$\frac{\Gamma \mid \bullet \vdash s : A^{-}}{\Gamma \mid \bullet \vdash \mathrm{refl}_s : \mathrm{Hom}(s, -s)}$$

Maintain the variables in two *context zones*: a groupoid **neutral zone** $\Gamma$ and a category **polar zone** $\Delta$

$$\Gamma \mid \Delta \vdash c : C$$

$s$ must be functorial with respect to the polar variables $x : A$ in $\Delta$, but only functorial with respect to the *core groupoid* of the neutral variables $y :: A$ in $\Gamma$.

**Key point** In a *neutral context*, one of the form $\Gamma \mid \bullet$, we can *coerce* between $A$ and $A^{-}$:

$$\frac{\Gamma \mid \bullet \vdash s : A^{-}}{\Gamma \mid \bullet \vdash -s : A} \qquad \frac{\Gamma \mid \bullet \vdash t : A}{\Gamma \mid \bullet \vdash -t : A^{-}} \qquad \frac{}{- - s \equiv s} \qquad \frac{}{- - t \equiv t}$$

# Composition and transport

$$\frac{\Gamma \mid \bullet \vdash A\ \textbf{type}}{\Gamma, y :: A \mid x \colon A^{-}, z \colon A, u \colon \mathrm{Hom}(x, y), v \colon \mathrm{Hom}(y, z) \vdash u \cdot v \colon \mathrm{Hom}(x, z)}$$

$$\frac{\Gamma \mid z \colon A \vdash P(z)\ \textbf{type} \quad \Gamma \mid \bullet \vdash p_0 \colon P(-s)}{\Gamma \mid z \colon A, u \colon \mathrm{Hom}(s, z) \vdash \mathrm{tr}_P^{+}\ u\ p_0 \colon P(z)}$$

$$\frac{\Gamma \mid x \colon A^{-} \vdash Q(x)\ \textbf{type} \quad \Gamma \mid \bullet \vdash q_0 \colon Q(-t)}{\Gamma \mid x \colon A^{-}, v \colon \mathrm{Hom}(x, t) \vdash \mathrm{tr}_Q^{-}\ v\ q_0 \colon Q(x)}$$

Between any two parallel morphisms, we have the *identity type*, which is the proposition that they are equal, represented as a type:

$$\frac{\Gamma \mid \Delta \vdash f\colon \mathsf{Hom}(s,t) \quad \Gamma \mid \Delta \vdash f'\colon \mathsf{Hom}(s,t)}{\Gamma \mid \Delta \vdash \mathsf{Id}(f,f')\ \textbf{type}}$$

(this is actually the Hom-type $\mathsf{Hom}_{\mathsf{Hom}(s,t)}(f,f')$, but we assert that $\mathsf{Hom}_{\mathsf{Hom}(s,t)}$ is an equivalence relation, not a general category). We can prove the category laws, e.g. that there's always a term of type

$$\mathsf{Id}(f \cdot (g \cdot h), (f \cdot g) \cdot h).$$

# 1 Synthetic-(Co)Inductive Category Theory

**Inductive Type** Define $A$ by *constructors* and an induction principle: in order to prove/construct something for all $x \colon A$, it suffices to do so just on the constructors.
E.g. $\mathbb{N}$ with constructors *zero* and *succ*.

**Coinductive Type** Define $A$ by *destructors* and a *coinduction* principle: for any bisimulation $R$ (a relation which is preserved under destructors), if $R(x, y)$ then $x = y$
E.g. $\mathrm{Stream}(X)$ with destructors $\mathrm{hd} \colon \mathrm{Stream}(X) \to X$ and $\mathrm{tl} \colon \mathrm{Stream}(X) \to \mathrm{Stream}(X)$. A bisimulation is a relation $R$ such that, for all $x, y$ such that $R(x, y)$,

$$\mathrm{hd}(x) = \mathrm{hd}(y) \qquad \text{and} \qquad R(\mathrm{tl}\ x, \mathrm{tl}\ y).$$

**Observation:**

Universal mapping properties are principles of induction

# Example: Products

Work in an arbitrary neutral context $\Gamma \mid \bullet$.Given $s, t \colon A$, say that the data

$$P \colon A \qquad \pi_1 \colon \mathrm{Hom}(-P, s) \qquad \pi_2 \colon \mathrm{Hom}(-P, t)$$

is a **product** of $s$ and $t$ if it satisfies the following principle of induction

$$\frac{\Gamma \mid x \colon A^-, u \colon \mathrm{Hom}(x, s), v \colon \mathrm{Hom}(x, t) \vdash M(x, u, v)\ \textbf{type} \qquad \Gamma \mid \bullet \vdash m \colon M(-P, \pi_1, \pi_2)}{\Gamma \mid x \colon A^-, u \colon \mathrm{Hom}(x, s), v \colon \mathrm{Hom}(x, t) \vdash \mathrm{elim}\ m\ (x, u, v) \colon M(x, u, v)}$$

Get the induced morphism:

$$\Gamma \mid x, u, v \vdash \langle u, v\rangle := \mathsf{elim}\ \mathsf{refl}_P \colon \mathsf{Hom}(x, -P)$$

the commutativity of the triangles:

$$\Gamma \mid x, u, v \vdash \mathsf{elim}\ \mathsf{refl}_{\mathsf{refl}_P} \colon \mathsf{Id}(\langle u, v\rangle \cdot \pi_1, u)$$

and so on.

Developed in this style ([Neu25a, Chapter 4]):

- Products and coproducts
- Pullbacks and pushouts
- Right and left adjoints (exponentials, all (co)limits of a given shape)

Not included there, but also have worked out:

- Initial and terminal objects
- Equalizers and coequalizers

A **right adjoint** of $F\colon A \to B$ consists of

$$U\colon B \to A \qquad\qquad \epsilon\colon \mathrm{Hom}(-(F \circ U), I_B)$$

such that

$$\frac{\Gamma, y :: B \mid x\colon A^-, u\colon \mathrm{Hom}(F(x), y) \vdash M(y, x, u)\ \textbf{type} \qquad \Gamma, y :: B \mid \bullet \vdash m\colon M(y, -U(y), \epsilon @ y)}{\Gamma, y :: B \mid x\colon A^-, u\colon \mathrm{Hom}(F(x), y) \vdash \mathrm{elim}\ m\ (y, x, u)\colon M(y, x, u)}$$

# What category-theoretic concepts are coinductive in nature?

**Coinduction** Any *bisimulation* (a relation that is preserved by the destructors) is contained within the diagonal relation (bisimilarity implies equality).

A morphism $f : \mathrm{Hom}(s, t)$ is a **monomorphism** if it satisfies the following coinduction principle

$$\frac{\Gamma \mid x : A^{-}, u : \mathrm{Hom}(z, -s), v : \mathrm{Hom}(z, -s) \vdash R(x, u, v)\ \textbf{type} \qquad \Gamma \mid x, u, v, r : R(x, u, v) \vdash \mathrm{bisim}_R(r) : \mathrm{Id}(u \cdot f, v \cdot f)}{\Gamma \mid x, u, v, r \vdash \mathrm{coind}_R\ r : \mathrm{Id}(u, v)}$$

# Future directions

- Better examples of coinductive concepts
- Relax polarity calculus to be able to write natural transformations componentwise
- Implement language, to allow for computer-formalized category theory

[Neu25a] Jacob Neumann.
*A Generalized Algebraic Theory of Directed Equality*.
PhD thesis, University of Nottingham, 2025.

[Neu25b] Jacob Neumann.
A judgmental construction of directed type theory, 2025.

[Shu18] Michael Shulman.
Brouwerś fixed-point theorem in real-cohesive homotopy type theory.
*Mathematical Structures in Computer Science*, 28(6):856–941, 2018.

# Thank you!

arxiv.org/abs/2510.17494
jacobneu.phd