A Type Theory for Synthetic 1-Category Theory

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CT Octoberfest 2024
27 October 2024

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MLTT Identity Types Given any type A and terms $t, t' \colon A$, we can form the type

$$\mathsf{Id}(t, t')$$

of *identities* between t and t'.

$$\frac{t \colon A}{\mathsf{refl}_t \colon \mathsf{Id}(t, t)} \qquad \frac{p \colon \mathsf{Id}(t, t')}{p^{-1} \colon \mathsf{Id}(t', t)} \qquad \frac{p \colon \mathsf{Id}(t, t') \quad q \colon \mathsf{Id}(t', t'')}{p \cdot q \colon \mathsf{Id}(t, t'')}$$

Idea: Modify this to synthetic *category* theory

MLTT Identity Types Given any type A and terms $t, t' : A$, we can form the type

$$\mathrm{Hom}(t, t')$$

of *morphisms* between t and t'.

$$\frac{t : A}{\mathrm{refl}_t : \mathrm{Hom}(t, t)} \qquad \frac{p : \mathrm{Hom}(t, t') \quad q : \mathrm{Hom}(t', t'')}{p \cdot q : \mathrm{Hom}(t, t'')}$$

Goal: A *Directed Type Theory* which

- Provides a language for synthetic *category* theory
- Has syntactic discipline to prevent symmetry from being provable
- Allows for *informal type theory*

Approaches to directed type theory

- Simplicial/cubical theories (hom types defined using directed interval)
  - [RS17], [KRW23], [GWB24], [Wei22]
- "2-dimensional" type theories
  - [LH11], [Nuy15]
  - [ANvdW23]
  - [NL23]
- Modal typing disciplines
  - [Nor19]
  - This theory

A word about semantics

- Use the formalism of *categories with families (CwFs)*; extend to *(1,1)-directed CwFs*, which automatically have an initial/syntax model
- Groupoid model [HS95] replaced with Category model; Setoid model [Hof95] replaced with Preorder model
- Instance of general construction [KKA19] to turn the (displayed) algebras of any GAT into a model of type theory

Opposites and neutral types

For every type (synthetic category) A, there is a type A^-, its **opposite**. The opposite-taking operation is an involution: $(A^-)^- = A$.

We can judge a type X to be **neutral**, meaning that it is a synthetic groupoid. The opposite of X will be isomorphic to X.

$$\frac{\Gamma : \mathsf{Con} \quad A : \mathsf{Ty}\,\Gamma}{A^- : \mathsf{Ty}\,\Gamma}$$

$$\frac{}{(A^-)^- = A}$$

$$X : \mathsf{NeutTy}\,\Gamma$$

Hom type former

For any type A and objects $t : A^-$ and $t' : A$, we can form the type

$$\mathrm{Hom}(t, t')$$

of A-morphisms from t to t'.

- Can iterate Hom, so *a priori* we're encoding higher-categorical structure
- The variance annotations will prevent us from proving symmetry for Hom (except for neutral types)
- If A is a neutral type, write $\mathrm{Id}(t, t')$ instead of $\mathrm{Hom}(t, t')$
- $\mathrm{Hom}_A(t, t')$ is isomorphic to $\mathrm{Hom}_{A^-}(t', t)$

Problem How can we type the identity hom, $\mathrm{refl} : \mathrm{Hom}(t, t)$?

Solution: Adopt a substructural neutrality-polarity calculus for *contexts*

Context variance

$$(-)^{-} : \mathsf{Con} \to \mathsf{Con}$$

$$\mathsf{NeutCon} \hookrightarrow \mathsf{Con}$$

$$\Gamma \cong \Gamma^{-} \text{ for all } \Gamma : \mathsf{NeutCon}.$$

$$\frac{\Gamma : \mathsf{NeutCon} \quad t' : \mathsf{Tm}(\Gamma, A)}{-t' : \mathsf{Tm}(\Gamma, A^{-})}$$

$$\frac{\Gamma : \mathsf{NeutCon} \quad t : \mathsf{Tm}(\Gamma, A^{-})}{-t : \mathsf{Tm}(\Gamma, A)}$$

$$\frac{\Gamma : \mathsf{NeutCon} \quad A : \mathsf{NeutTy}\,\Gamma}{\Gamma \triangleright A : \mathsf{NeutCon}}$$

When working informally, our ambient context is always assumed to be neutral.

Given a *closed term (w.r.t. the current context)* $t : A^{-}$, we get $-t : A$. Given a *term* $t' : A$, we get a term $-t' : A^{-}$.

Variable Negation Rule An expression can be negated only if all the variables it contains are of neutral types

Directed path induction in neutral contexts

Every term $t : A^-$ comes equipped with a term

$$\mathrm{refl} : \mathrm{Hom}(t, -t).$$

Principle of Directed Path Induction Suppose $t : A^-$ and $M(x, y)$ is a type family depending on variables $x : A$ and $y : \mathrm{Hom}(t, x)$. Then, given

$$m : M(-t, \mathrm{refl}),$$

there is a term

$$\mathrm{ind}_M(m, x, y) : M(x, y)$$

for all x, y.

Truncation

We stipulate that our synthetic categories are synthetic *(1,1)-categories*:

- each type $\mathsf{Hom}(t, t')$ is neutral
- the *uniqueness of identity proofs* holds for identities of homs: if $\alpha \colon \mathsf{Id}(p, q)^{-}$ and $\beta \colon \mathsf{Id}(p, q)$ for $p \colon \mathsf{Hom}(t, t')^{-}$ and $q \colon \mathsf{Hom}(t, t')$, then we have an identity

$$\mathsf{Id}(\alpha, \beta).$$

- Define the composition $p \cdot q \colon \mathsf{Hom}(t, t'')$ by directed path induction:
  $p \cdot \mathsf{refl} = p$.
- Get an identity $\mathsf{Id}(\mathsf{refl} \cdot q, q)$ by directed path induction on q:
  $\mathsf{refl}_{\mathsf{refl}} : \mathsf{Id}(\mathsf{refl} \cdot \mathsf{refl}, \mathsf{refl})$.
- Get an identity $\mathsf{Id}(p \cdot (q \cdot r), (p \cdot q) \cdot r)$ by directed path induction on r:

$$\mathsf{refl}_{p \cdot q} \colon \mathsf{Id}(p \cdot (q \cdot \mathsf{refl}), (p \cdot q) \cdot \mathsf{refl})$$

Key point: These hold automatically for every type we can express in the theory. We never have to *prove something is a category*.

$C : \{t : \mathrm{Tm}(\Gamma, A^-)\} \to \mathrm{Ty}\ (\Gamma \triangleright^+ A \triangleright^+ \mathrm{Hom}(t'[p_A], v))$
$C = \mathrm{Hom}(t[p_A], v_A)$

$\_\cdot\_ : \{t\ t' : \mathrm{Tm}(\Gamma, A^-)\}\{t'' : \mathrm{Tm}(\Gamma, A)\}$
$\quad \to \mathrm{Tm}(\Gamma, \mathrm{Hom}(t,-t')) \to \mathrm{Tm}(\Gamma, \mathrm{Hom}(t',t'')) \to \mathrm{Tm}(\Gamma, \mathrm{Hom}(t,t''))$
$p \cdot q = (J_{t',C}\ p)\ [t'',q]$

$r\text{−unit} : (q : \mathrm{Tm}(\Gamma, \mathrm{Hom}(t',t''))) \to \mathrm{Tm}(\Gamma, \mathrm{Id}(\mathrm{refl}_{t'} \cdot q, q))$
$r\text{−unit}\ q = (J_{t',R}\ \mathrm{refl}_{\mathrm{refl}})[t'',q]$ **where**
$\quad R : \mathrm{Ty}\ (\Gamma \triangleright^+ A \triangleright^+ \mathrm{Hom}(t'[p_A], v_A))$
$\quad R = \mathrm{Id}((J_{t',C}\ \mathrm{refl}_{t'}), v_{\mathrm{Hom}(t'[p_A],v_A)})$
$l\text{−unit} : (p : \mathrm{Tm}(\Gamma, \mathrm{Hom}(t,-t'))) \to \mathrm{Tm}(\Gamma, \mathrm{Id}(p \cdot \mathrm{refl}_{t'}, p))$
$l\text{−unit}\ p = \mathrm{refl}_p$
$\mathrm{assoc} : (p : \mathrm{Tm}(\Gamma, \mathrm{Hom}(t,-t'))) \to (q : \mathrm{Tm}(\Gamma, \mathrm{Hom}(t',-t'')))$
$\quad \to (r : \mathrm{Tm}(\Gamma, \mathrm{Hom}(t'',t'''))) \to \mathrm{Tm}(\Gamma, \mathrm{Id}(p \cdot (q \cdot r), (p \cdot q) \cdot r))$
$\mathrm{assoc}\ p\ q\ r = (J_{t',S}\ \mathrm{refl}_{p \cdot q})[t''',r]$ **where**
$\quad S : \mathrm{Ty}\ (\Gamma \triangleright^+ A \triangleright^+ \mathrm{Hom}(t[p_A], v_A))$
$\quad S = \mathrm{Id}((J_{t,C}\ (p \cdot q)), v_{\mathrm{Hom}(t[p_A],v_A)})$

Synthetic functors

We can define functions $f\colon A \to B$ in our theory, but apply them to terms $t\colon A^{-}$.

These behave like synthetic functors: we have an operation map f sending $p\colon \mathrm{Hom}(t, t')$ to

$$\text{map } f\ p\colon \mathrm{Hom}(-f(t), f(-t')),$$

defined by directed path induction: map f $\mathrm{refl}_t = \mathrm{refl}_{-f(t)}$.

Exercise Prove that this morphism part is functorial with respect to the synthetic category structure

Exercise Construct an identity between map $(g \circ f)\ p$ and map g (map $f\ p$)

Variable negation rule guards symmetry

If A is a neutral type, then, given $t : A^-$, we can form the type family

$$S(x, y) := \mathsf{Id}(-x, -t).$$

depending on variables $x : A$ and $y : \mathsf{Id}(t, x)$. Since $\mathsf{refl} : \mathsf{Id}(t, -t)$, i.e. $\mathsf{refl} : S(-t, \mathsf{refl})$, we get

$$\mathsf{ind}_S(x, y) \quad : \mathsf{Id}(-x, -t)$$

So, given a particular $t' : A$ and $p : \mathsf{Id}(t, t')$, we can construct

$$p^{-1} := \mathsf{ind}_S(t', p) \quad : \mathsf{Id}(-t', -t).$$

Question Why does this only work for A neutral?

Current work

- Natural transforms
- More category theory ((co)limits, adjoints, Yoneda, . . .)
- Universe of sets
- Synthetic (2,1)-category theory with a universe of categories
- Formalization

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Thank you!