A Sampling of Synthetic 1-Category Theory

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Big Idea

Path Induction

Yoneda Lemma

Directed TT
in the Category Model

Univalence

Free Theorems
(Parametricity)

Warning: Work in progress

Preprint:
jacobneu.github.io/HoTT-UF-2024/preprint

0 Setting: The Category Model

[HS95] define the **groupoid model of type theory** as a model of type theory (a CwF) with

- $\mathsf{Con} = \mathsf{Grpd}$
- $\mathsf{Ty}\ \Gamma = [\Gamma, \mathsf{Grpd}]$
- $\ldots$

We **polarize** this model, obtaining the **category model of type theory**,

- $\mathsf{Con} = \mathsf{Cat}$
- $\mathsf{Ty}\ \Gamma = [\Gamma, \mathsf{Cat}]$
- $\ldots$

Goal: Develop this as a model of directed type theory and do synthetic (1-)category theory

The *opposite* operation on categories furnishes us with two 'negation' operations, one on **contexts** (with similar rules as [LH11]):

$$\frac{\Gamma : \mathsf{Con}}{\Gamma^- : \mathsf{Con}} \qquad \frac{\gamma : \mathsf{Sub}\ \Delta\ \Gamma}{\gamma^- : \mathsf{Sub}\ (\Delta^-)\ (\Gamma^-)} \qquad \mathsf{Cat} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}$$

and one on **types** (same rule as in [Nor19]):

$$\frac{A : \mathsf{Ty}\ \Gamma}{A^- : \mathsf{Ty}\ \Gamma} \qquad \Gamma \xrightarrow{A} \mathsf{Cat} \xrightarrow{(-)^{\mathrm{op}}} \mathsf{Cat}$$

These are tied together by the **negative context extension** operation:

$$\frac{\Gamma : \mathsf{Con} \quad A : \mathsf{Ty}(\Gamma^-)}{\Gamma \rhd^- A : \mathsf{Con}} \qquad \mathsf{Sub}\ \Delta\ (\Gamma \rhd^- A) \cong \sum_{\gamma : \mathsf{Sub}\ \Delta\ \Gamma} \mathsf{Tm}(\Delta^-, A[\gamma^-]^-)$$

The deep polarization allows us to formulate Π-types (also following [LH11]):

$$\frac{A \colon \mathrm{Ty}(\Gamma^{-}) \qquad B \colon \mathrm{Ty}(\Gamma \triangleright^{-} A)}{\Pi(A, B) \colon \mathrm{Ty}\,\Gamma}$$

$$app \quad : \quad \mathrm{Tm}(\Gamma, \Pi(A, B)) \cong \mathrm{Tm}(\Gamma \triangleright^{-} A, B) \quad : \quad \lambda$$

Hom types

Adapting the definition of identity types in the groupoid model, we get **hom types**

$$\frac{A\colon \mathrm{Ty}\,\Gamma \quad t\colon \mathrm{Tm}(\Gamma, A^{-}) \quad t'\colon \mathrm{Tm}(\Gamma, A)}{\mathrm{Hom}_A(t, t') : \mathrm{Ty}\,\Gamma}$$

(note the polarities).

- Hom types are not symmetric (in general): in the empty context (where types are categories and terms are objects), we can come up with A, t, t' such that $\mathrm{Hom}_A(t, t')$ is inhabited but $\mathrm{Hom}_A(t', t)$ isn't.
- Note: hom types can be iterated. In the category model, homs between homs *are* symmetric and unique, i.e. form an equivalence relation. We'll denote this $\mathrm{Id}(,)$.

1 Directed Path Induction

Idea: if it works for refl, it works for all

Recall the rules for introducing and eliminating identity types:

$$\frac{t\colon \mathsf{Tm}(\Gamma, A)}{\mathsf{refl}\colon \mathsf{Tm}(\Gamma, \mathsf{Id}_A(t, t))} \qquad \frac{\begin{array}{c} t\colon \mathsf{Tm}(\Gamma, A) \\ M\colon \mathsf{Ty}(\Gamma \triangleright (z : A) \triangleright \mathsf{Id}_A(t, z)) \\ m\colon \mathsf{Tm}(\Gamma, M[t, \mathsf{refl}]) \\ t'\colon \mathsf{Tm}(\Gamma, A) \\ p\colon \mathsf{Id}_A(t, t') \end{array}}{J_M\ m\ t\ p\colon \mathsf{Tm}(\Gamma, M[t', p])}$$

Goal: *Directed path induction* for hom types

Problem: How do we type refl?

Refl requires bi-variant terms

$$\frac{t : \mathrm{Tm}(\Gamma, A)}{\mathrm{refl} : \mathrm{Tm}(\Gamma, \mathrm{Hom}_A(t, t))}$$

Refl requires bi-variant terms

$$\frac{t : \mathrm{Tm}(\Gamma, A^{-})}{\mathrm{refl} : \mathrm{Tm}(\Gamma, \mathrm{Hom}_A(t, t))}$$

How do we make t both positive and negative?

[Nor19] gets around this by using *core types*, which can also be interpreted in the category model:

$$\frac{A\colon \mathrm{Ty}\,\Gamma}{A^0\colon \mathrm{Ty}\,\Gamma} \qquad \Gamma \xrightarrow{A} \mathrm{Cat} \xrightarrow{\mathrm{core}} \mathrm{Grpd} \hookrightarrow \mathrm{Cat}$$

A term $t\colon \mathrm{Tm}(\Gamma, A^0)$ can be turned into either a term of type A^- or A, allowing us to introduce refl and state directed path induction.

Problem: This only allows us to prove things about homs *based at a term of type* A^0, not arbitrary homs.

Solution 2: Neutral *contexts* and Coercion

Our solution is to instead work in **neutral contexts**, i.e. groupoids.In a neutral context, we can coerce between A and A^-:

$$\frac{\Gamma : \mathsf{NeutCon} \quad a : \mathsf{Tm}(\Gamma, A^s)}{-a : \mathsf{Tm}(\Gamma, A^{-s})}$$

$$\frac{t : \mathsf{Tm}(\Gamma, A^-)}{\mathsf{refl}_t : \mathsf{Tm}(\Gamma, \mathsf{Hom}_A(t, -t))} \qquad \frac{t' : \mathsf{Tm}(\Gamma, A)}{\mathsf{refl}_{t'} : \mathsf{Tm}(\Gamma, \mathsf{Hom}_A(-t', t))}$$

Note: Neutral contexts don't force symmetry

(counterexample was in empty context, which is neutral)

Directed Path Induction

$$\frac{\begin{array}{c} t : \mathrm{Tm}(\Gamma, A^{-}) \\ M : \mathrm{Ty}(\Gamma \triangleright^{+} (z : A) \triangleright^{+} \mathrm{Hom}_A(t, z)) \\ m : \mathrm{Tm}(\Gamma, M[-t, \mathrm{refl}_t]) \end{array} \qquad \begin{array}{c} t' : \mathrm{Tm}(\Gamma, A) \\ p : \mathrm{Tm}(\Gamma, \mathrm{Hom}_A(t, t')) \end{array}}{\mathrm{J}^{+}_{M}\ m\ t'\ p : \mathrm{Tm}(\Gamma, M[t', p])}$$

Example: Composition

Given $t, t' \colon \mathrm{Tm}(\Gamma, A^-)$ and $t'' \colon \mathrm{Tm}(\Gamma, A)$ with homs $p \colon \mathrm{Hom}_A(t, t')$ and $q \colon \mathrm{Hom}_A(-t', t'')$, we can define $p \cdot q \colon \mathrm{Hom}_A(t, t'')$ by directed path induction on q:

$$p \cdot \mathrm{refl} = p.$$

- Can prove associativity (up to identity types between homs) by directed path induction
- One unit law, $p \cdot \mathrm{refl} = p$, holds definitionally, other provable by directed path induction on p.

2 Connections

Connection #1: the Dependent Yoneda Lemma(s)

(inspired by [RS17])

The (covariant) Dependent Yoneda Lemma

Lemma For any $F : \mathbb{C} \to \mathrm{Set}$ and $G : (\int F) \to \mathrm{Set}$, there is an isomorphism

$$G(I, \phi) \quad \cong \quad \int_{J:\mathbb{C}} (j : \mathbb{C}(I, J)) \to G(J, F\, j\, \phi)$$

natural in (I, ϕ).

Instantiate for $F = \mathrm{Hom}(I, -)$ and $\phi = \mathrm{id}_I$:

$$G(I, \mathrm{id}_I) \quad \cong \quad \int_{J:\mathbb{C}} (j : \mathbb{C}(I, J)) \to G(J, j \circ \mathrm{id}_I)$$

$$G(I, \mathrm{id}_I) \underset{\mathsf{ev\_id}}{\overset{(*)}{\rightleftarrows}} \int_{J:\mathbb{C}} (j : \mathbb{C}(I, J)) \to G(J, j)$$

$$M[-t, \mathrm{refl}_t] \underset{\mathsf{ev\_refl}}{\overset{\mathsf{J}^+_M}{\rightleftarrows}} \int_{t':A} (p : \mathrm{Hom}_A(t, t')) \to M[t', p])$$

Connection #2:
(Trunctated) Directed Univalence

Universe of Sets in the Category Model

We have $\mathbf{U} \colon \mathrm{Ty}\ \bullet$, given by the category Set. For each $X \colon \mathrm{Tm}(\bullet, \mathbf{U})$, we get $\mathrm{El}(X) \colon \mathrm{Ty}\ \bullet$, which is interpreted as the discrete category on X.

Given sets X, Y, i.e. $X, Y \colon \mathrm{Tm}(\bullet, \mathbf{U})$, the hom type $\mathrm{Hom}_{\mathbf{U}}(X, Y) : \mathrm{Ty}\bullet$ is interpreted as the discrete category on the set $X \to Y$.

Truncated Directed Univalence in the Category Model

Given sets X, Y, i.e. $X, Y : \mathsf{Tm}(\bullet, \mathbf{U})$, the hom type $\mathsf{Hom}_{\mathbf{U}}(X, Y) : \mathsf{Ty}\bullet$ is interpreted as the discrete category on the set $X \to Y$.

$$\begin{aligned} \mathsf{Tm}(\bullet, \mathsf{El}(X) \to \mathsf{El}(Y)) &\cong \mathsf{Tm}(\mathsf{El}(X), \mathsf{El}(Y)) \\ &\cong X \to Y \end{aligned}$$

We can internalize this equivalence between $\mathsf{Hom}_{\mathbf{U}}(X, Y)$ and $\mathsf{El}(X) \to \mathsf{El}(Y)$:

$$\begin{aligned} \text{hom-to-func} &: \mathsf{Tm}(\bullet, \mathsf{Hom}_{\mathbf{U}}(X, Y) \to (\mathsf{El}(X) \to \mathsf{El}(Y))) \\ \text{func-to-hom} &: \mathsf{Tm}(\bullet, (\mathsf{El}(X) \to \mathsf{El}(Y)) \to \mathsf{Hom}_{\mathbf{U}}(X, Y)) \end{aligned}$$

Note that hom-to-func can be defined by directed path induction.

Future work: Un-truncated version

Connection #3: Naturality for Free!

Given $C, D\colon \mathsf{Ty}\ \bullet$ and $f\colon \mathsf{Tm}(\bullet, C \to D)$, define for a given $c\colon \mathsf{Tm}(\bullet, C^{-})$ and $c'\colon \mathsf{Tm}(\bullet, C)$

$$\mathsf{map}_f\colon \mathsf{Tm}(\bullet, \mathsf{Hom}_C(c, c')) \to \mathsf{Tm}(\bullet, \mathsf{Hom}_D((f\$c), (f\$c')))$$

(where $f\$c = (app\ f)[c]\colon \mathsf{Tm}(\Gamma, D)$) by directed path induction:

$$\mathsf{map}_f\ \mathsf{refl}_c = \mathsf{refl}_{f\$c}\colon \mathsf{Tm}(\bullet, \mathsf{Hom}_D((f\$c), (f\$c))).$$

Naturality for free!

Given another $g\colon \mathsf{Tm}(\bullet, C \to D)$ and any $\alpha\colon \mathsf{Tm}(\bullet, \Pi(c : C, \mathsf{Hom}_D((f\$c), (g\$c))))$, we can construct a term of type

$$\mathsf{Id}_{\mathsf{Hom}_D((f\$c),(g\$c'))}\left(\alpha_c \cdot (\mathsf{map}_g p), (\mathsf{map}_f p) \cdot \alpha_{c'}\right)$$

$$\begin{array}{ccc} f\$c & \xrightarrow{\alpha_c} & g\$c \\ \downarrow{\scriptstyle \mathsf{map}_f\ p} & & \downarrow{\scriptstyle \mathsf{map}_g\ p} \\ f\$c' & \xrightarrow[\alpha_{c'}]{} & g\$c' \end{array}$$

Again by directed path induction, on p:

$$\mathsf{map}_g\ \mathsf{refl} = \mathsf{refl}$$
$$\alpha_c \cdot \mathsf{refl} = \alpha_c$$
$$\mathsf{map}_f\ \mathsf{refl} = \mathsf{refl}$$

Future work: More free theorems

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Thank you!

jacobneu.github.io/HoTT-UF-2024/preprint
jacobneu.github.io/HoTT-UF-2024/slides