# A Generalized Algebraic Theory of Directed Equality

Jacob Neumann

PhD Thesis Defense

16 September 2025

University of Nottingham

# Outline

- Research Questions
- Main contributions

jacobneu.com/defense

RQ1

What is an appropriate methodology for developing directed type theory and synthetic category theory?

# Key Points

- Generalized Algebra
  - Canonical examples of GATs (presented in NOUGAT syntax)
  - Rehearse KKA's treatment of QIITs, restricted to GATs
    - ONEGAT language
    - Quotient inductive-inductive definitions of (displayed) $\mathfrak{G}$-algebras, $\mathfrak{G}$-homomorphism, ..., initial $\mathfrak{G}$-algebra
    - Concrete CwFs
  - Fibrancy and "the Looking-Glass question"
- Extensions to the GAT $\mathfrak{CwF}$, justified by the category model
- Informal *synthetic* category theory "formalized" into the syntax of (1,1)-directed type theory and interpreted into *analytic* category theory (closed types in the category model)

# No. 0: The Looking-Glass Question

Fibrancy

Cosmology

Auto-Synthesis

RQ2

How can the groupoid model's uses of symmetry be made explicit in the syntax of type theory?

# Polarity Problems

- **Shallow polarity problems** (e.g. $t$ in $\mathrm{Hom}(t, t')$)
- **Deep polarity problems** (e.g. $A$ in $\Pi(A, B)$)
- **Divariant (based)** (e.g. $t$ in $\mathrm{refl}_t$)
  - Divariance problem with refl can be solved with core types (North), but this doesn't seem to address the difficulties with Π-types
  - Proposed solution: require $t$ to be a term in a neutral context
- **Divariant (variable)** (e.g. $x$ in $\mathrm{Hom}(F(x), G(x))$)
  - Not solved here (though I have some ideas!)

# No. 1: Separating shallow and deep polarity

# No. 2: Neutral Contexts

...for polarized Π-types

...as a solution to the "refl divariance problem"

RQ3

# What is directed equality, and how does it work?

# No. 3: The polarity calculus guards J, preventing symmetry from being provable

(independence in practice)

## Interpretations of directed equality

- Synthetic Category Theory
- Directed Homotopy Theory
- Computational processes
- Temporal logic

## Phenomena to interpret

- refl
- Transport
- "Directed singleton contractibility", i.e. the initiality/terminality of refl among (co)slices
  - Would like to expand on this more in revised thesis
- (Failure of) Symmetry
- (Failure of) the Uniqueness of Homs principle

RQ4

What does synthetic category-theoretic reasoning in the directed type theory of the category model look like?

- **Informal Directed Type Theory**
  - Distinction of terms/types which are "closed" (w.r.t the ambient neutral context) vs. "in-telescope"
- **Synthetic-Inductive Category Theory**
- Limitations of current theory:
  - Pointwise natural transforms and directed function extensionality, Yoneda Lemma
  - (Co)Limits of a given, infinite diagram
  - Internalize some of the quantification

# No. 4: Informally working in a neutral context

# No. 5: Universal Mapping Properties as Principles of Induction

# Hopes for the future

- Further study into generalized algebra and the semantics of type theory
- Further study into Polarized and Neutral-Polarized CwFs
- Further development of (1,1)-directed type theory
- Further study of $(m, n)$-directed type theory
- Connection to other styles of directed type theory
- Further development of synthetic category theory

# Fin